2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 8 – July 19, 2012

Vector Valued Functions

- Warm up exercise:

- Often functions depend on more than one variable.

  Examples:

  (i) The altitude (output variable) depends of latitude and longitude (two input variables).

  (ii) A magnetic force depends on the position of the point in the plane. Input variables – (x, y), the coordinates of the point. Output variables (a, b), the horizontal and vertical components of the force.

- Transformations T in the plane take one point in the plane, (x, y), and map it to a different point in the plane $(f(x, y), g(x, y))$. Thus $T(x, y) = (f(x, y), g(x, y))$.

  Examples:

  1) Reflection across the x - axis: $T(x, y) = (x, -y)$.

  2) Rotation by $45°$ about the origin: $T(x, y) = (\frac{\sqrt{2}}{2}(x - y), \frac{\sqrt{2}}{2}(x + y))$.

- Necessary condition for $T(x, y) = (f(x, y), g(x, y))$ to be a rigid motion: the functions $f(x, y)$ and $g(x, y)$ must be linear. This condition is not sufficient. For example $T(x, y) = (x+y+3, y-1)$ is not a rigid motion (look what it does to the triangle with vertices at $(0, 0), (1, 0), (1, 2)$).

- Rewrite $T(x, y)$ using matrices:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

- Operations with matrices: addition, scalar multiplication (multiplying a matrix by a number), matrix multiplication, identity matrix, inverse of a matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a+A & b+B \\ c+C & d+D \end{pmatrix}$$

$$n \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} na & nb \\ nc & nd \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{pmatrix}$$

The identity matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

provided $\mathbf{ad - bc \neq 0}$.

- Isometry Theorem: $T(x, y)$ is an isometry (*i.e.*, rigid motion) if and only if there is an orthogonal matrix (see definition below) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ such that

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Definition: A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is orthogonal if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$