2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 5 – July 16, 2012

End Behavior of Functions, Rates of Growth, Exponential Functions

- **End Behavior**
  - We've already mentioned it briefly when talking about rational functions.
  - What do you mean by end behavior/how do you think about it? (Ask again.)
  - Key idea: **Bound for a function**
    - A number M is an **upper bound** for a function if the *values* of f are always less than or equal to M.
    - An upper bound is similar to a maximum, but not the same. Why?
    - Examples
      - 1 is an upper bound for $\sin(x)$, as is 1.2, 3, 99, 2^{10}
      - $\sin(x)$ has a maximum on $[0, \pi]$ or $[0, \pi/2]$ but not $[0, \pi/2)$.
      - $5x^2$ has no upper bound
      - -1, 0, 73, 1, 5^{72} are upper bounds for $-1 - x^2$
      - odd degree polynomials have no upper bound
      - $\log(x)$ has no upper bound (any logarithm)
      - Exponentials have no upper bound 2^x
      - An upper bound for the age of a human is 150 years
    - Upper bounds are useful for making estimates. The height of levees of a river are presumably upper bounds on the river level
    - In later courses, we talk about the least upper bound of a function. Maximum are least upper bounds, but not the converse.
    - A number M is a **lower bound** for a function if the *values* of f are always greater than or equal to M.
    - An lower bound is similar to a minimum, but not the same. Why?
    - Examples
      - a similar list to the above
      - odd degree polynomials have no lower bound
      - even degree polynomials with positive leading coefficient have lower bounds
    - The greatest lower bound is analogous to the least upper bound.
  - A function is **bounded** if it has an upper bound and a lower bounded, or a function is bounded if there is an $M \geq 0$, $|f(x)| \leq M$ for all x in the domain of f.
    - $\sin(x)$ and $\cos(x)$ are bounded by 1.
    - $51 + 17\sin(x)$ is bounded by 68.

    * polynomials are not bounded
  - **Infinite end behavior:**
    * A functions tends to ∞ for positive x if for any M, the values of f get above M and stay above M as x gets large
    * A functions tends to $-\infty$ for positive x if for any M, the values of f get below M and stay below M as x gets large
      · for polynomials, the sign of the leading term and the parity (even/odd) of the leading exponent determine as we known
      · for exponentials Ae^{kx} the signs of A and k determine the end behavior (A and k not zero)

Ae^{kx}	$k > 0$	$k < 0$
$A > 0$	∞ for $x > 0$	∞ for $x < 0$
$A < 0$	$-\infty$ for $x > 0$	$-\infty$ for $x < 0$

- **Horizontal asymptotes**
  - f has a horizontal asymptote at L as x becomes large if for any small positive number c, the values of f get closer to L than C and stay closer for sufficiently large values of x
    * $\sin(x)$ does not have $y = 1$ (or $y =$ any number for that matter) as a horizontal asymptote
    * The only polynomials with horizontal asymptotes are the constants, for example, $p(x) = 17$ or $p(x) = -\sqrt{2}$.
    * rational functions as we noted before
    * for exponentials Ae^{kx} the sign of k determines the end behavior (A and k not zero), A determines the approach

Ae^{kx}	$k > 0$	$k < 0$
$A > 0$	$y = 0$ for $x < 0$ from above	$y = 0$ for $x > 0$ from above
$A < 0$	$y = 0$ for $x < 0$ from below	$y = 0$ for $x > 0$ from below

  - Quotients of exponentials
    *
$$f(x) = \frac{3}{2 + e^{-x}}, \quad g(q) = \frac{17}{5e^{-2x} + 92}$$

- **Order of growth**
  - How fast do functions grow as x becomes large?
  - How do we compare?
    * **Consider the end behavior of the quotient.**
    * If f and g are positive functions for $x > 0$, the growth of f and g for positive x is:

- If $\frac{f(x)}{g(x)}$ has a horizontal asymptote at $y = 0$, g grows faster than f. (This is $f(x) = o(g(x))$.)
- If $\frac{f(x)}{g(x)}$ has infinite end behavior f grows faster than g. (This is $g(x) = o(f(x))$.)
- If $\frac{f(x)}{g(x)}$ has a horizontal asymptote at $y = c \neq 0$, f and g have the same rate of growth. (This is $f(x) = O(g(x))$.)

* So what about the following pairs, which grows faster:
  - x^2 and x^4?
  - $192x^3 + 73x^2 + 10^{45}$ and x^4?
  - x^{72} and 2^x?
  - $\log(x)$ and x^2?
  - $\sqrt{x}$ and $\log(x)$?
  - 6^x and 3^x?

Applications

- Importance of Exponentials and base e
- Population Growth
  - Exponential or Malthusian growth: (Sketch family of functions)
    * Whatever the population, in the same unit of time it increases by the same amount.
    $$P(t) = P_0 e^{rt}.$$
    * Notice, for example:
    $$P(34)/P(32) = P_0 e^{r34}/P_0 e^{r32} = e^{2r}$$
    * From what we've said above we know the end behavior for $r > 0$.
  - Logistic model. This is growth constrained by resources. It takes the form:
  $$P(t) = \frac{M}{1 + Ae^{-rt}}$$
- The following are scientific examples and demonstrate the types of problems we expect students to be able to solve.
- Radioactive Decay–End behavior is a horizontal asymptote at 0:
  - $A(t) = A_0 e^{-rt}$, $r > 0$.
  - Important quantity: The amount of time it takes half the amount to decay, $t_{1/2}$ defined by
  $$A(t_{1/2}) = \frac{1}{2}A_0 \Longrightarrow \frac{1}{2}A_0 = A_0 e^{-rt_{1/2}}$$

- For radioactive isotopes, half-life is a common description of how long they last:
  * Radioactive iodine, Iodine-131, or ${}^{131}I$: Half-life 8 days. Fukushima Daiichi nuclear disaster. (Iodine pills–saturating the thyroid with regular, non-radioactive iodine-127)
  * Radioactive caesium-137, half-life of 30.17 years. Can be treated by doses of Prussian blue, a pigment which caesium binds to.
- Problems we give typically provide two pieces of information so the formula A_0e^{-rt} can be found (we will see toomrrow). Then something is asked about the values.