2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 3 – July 11, 2012

Allometric Fitting: Oxygen consumption of organs.

Power Functions, Allometry and a dabble of others.

- Power functions:
  - One natural next step after polynomials–fractional powers of a single variable
  - A fairly subtle topic–mathematical development
    - ∗ Begin with powers of the form $\frac{1}{n}$, where n is a natural number.
      $$\sqrt{x} = x^{\frac{1}{2}}, \quad \sqrt[3]{x} = x^{\frac{1}{3}}, \ldots$$
    - ∗ We understand
      - · $y = x^{1/n}$ to mean that y is a number with the property that $y^n = x$.
      - · So, $y = 4096^{1/12}$ or $\sqrt[12]{4096} = 2$ because $2^{12} = 4096$.
    - ∗ There are some numbers whose roots we expect students to recognize, but generally remembering roots is of limited importance.
    - ∗ Meaning:
      - · This is really one of the first examples of inverse functions
      - · $x = g(y)$ is an inverse function of $y = f(x)$ if $y = f(g(y))$ and $x = g(f(x))$.
      - · We must pay attention to the domains of f and g.
      - · "horizontal line" test
      - · Reflecting a graph – how to teach it?
    - ∗ Graphs of $y = x^{1/n}$
      - · domains (even/odd distinction)
      - · family of functions – all pass through $(0, 0)$ and $(1, 1)$
      - · end behavior?
      - · as n increases, the graph flattens out, but no horizontal asymptote – not bounded
      - · rudimentary plotting
    - ∗ More general: $y = x^{m/n}$.
      - · Algebra key: $y = x^{m/n} = (x^{1/n})^m$.
      - · Plot powers of $x^{1/5}$.
    - ∗ More general yet, $y = x^r$, where r is a positive number that is not rational:
      $$y = x^{\sqrt{2}}$$
      - · Not so easy to think about.
      - · If we were really serious, $\sqrt{2} = 1.4142\ldots$ so we could consider
        $$x^{14/10}, \quad x^{141/100} \ldots$$
        Of course we never do

- Pause for an example:
  - What are possible applications?
    - The Pythagorean Theorem
    - Length of side of square is square root of area
    - Length of edge of cube is cube root of volume
    - etc.(?)
  - Hands-on application
    - Working with a simple pendulum, find the time needed for an oscillation as a function of the length of the pendulum.
    - Plot the results of all groups
    - See that it should be $T(L) = Ax^r$. From the graph, we expect $r = \frac{m}{n} < 1$.
    - A more sophisticated approach – logs
      - We use two points on the graph and logarithms base 10 to find A and r
      - $$y = Ax^r \longrightarrow \log(y) = \log(Ax^r) = \log(A) + r\log(x)$$
      - Why is this interesting? (we only have to solve a system of two linear equations)

  The data fits the curve $T(L) = 2\pi\sqrt{\frac{L}{g}}$, where g is the gravitational accelaration.

- Another application – Tsunamis
  - Recent 2011 Tohoku earthquake 5th most powerful recorded since 1900
  - Enormously powerful tsunami was generated (waves $\tilde{1}30$ ft!)
  - Also 2004 Sumatra earthquake set off tsunami killing 230,000 people
  - Tsunami waves are very long, so although the ocean is deep at most (6.78 miles deep at Marianas Trench) wavelengths of several hundred kilometers make them long waves–shallow water!
  - This is of interest because waves of this type have velocity given by
    $$v = \sqrt{gd}$$
    where g is the acceleration due to gravity and d is ocean depth.
  - Note the units $\sqrt{ft/sec^2 \cdot ft} = ft/sec.$
  - So the Pacific Ocean is mostly deeper than 3000 meters says, an tsunami moves at speeds in excess of:
    $$v = \sqrt{10 \cdot 3000} \approx 173m/s \approx 620km/hr$$
    As high as 950 km/hr!
  - Could cross the Pacific Ocean in approximately one day
  - This speed property is generally true for long waves in shallow water–famous standing waves or solitons

Allometry

- Allometry – study of the relationship of body size to shape, anatomy, physiology and finally behavior,
- Goes back to Otto Snell in 1892, D'Arcy Thompson in 1917 and Julian Huxley in 1932.
- My source: *Scaling: Why is Animal Size So Important* Knut Schmidt-Nielsen, Cambridge, 1984.
- The **range of animal sizes** P. 2.
- Geometric Scaling
  - Surface area is proportional to length squared
  - Volume is proportional to length cubed
  - surface area is proportional to volume to the 2/3 power
- Allometric Scaling
  - Animals don't scale geometrically across species
  - Certain proportions for animals do scale with size
- The *allometric equation*

  $$y = ax^b \text{ or } \log y = \log a + b \log x.$$

  Note b is the slope of the straight line in a logarithmic plot. Slope 1 is a simple proportion, $y = ax$.
- **Were dinosaurs stupid?** (p. 26-27)
  - Notice the slopes are quite similar for different groups
  - Notice the slope for all mammals is higher, why?
  - Dinosaurs fit naturally into the reptile group
  - If exponents the same, coefficients tell relative size of brain with respect to mass
- **Metabolic Rates (Oxygen Consumption) of Mammals**(p. 57)
  - Metabolic rate to body mass

    $$P_{met} = 73.3 M_b^{0.74}.$$

  - Look at the fit!

- **Group Exercise – Allometric Fitting**
- **The Meaning of Time**
  - Heartbeats/min
    - Shrew = 1000
    - Elephant = 30
  - frequency = 1/time
  - **frequency to body mass in kg**

    $$f_h = 241 M_b^{-0.25}$$

  - time per heartbeat in terms of mass

    $$t_h = \frac{1}{241} M_b^{0.25} = 0.249 M_b^{0.25}$$

    the last is in seconds. So for 1 kg, 1/4 second /beat, 4 beats/sec 240/min
  - The respiratory frequency is

    $$f_{resp} = 53.5 M_b^{-0.26}$$

    so we get

    $$f_h / f_{resp} \approx 4.5 M_b 0.1 \approx 4.5$$

    So across ALL organisms the ratio of heartbeats to breaths is 4.5!
- Length vs. Speed from Bonner
- THERE IS MUCH MORE!