2012 Summer Workshop, College of the Holy Cross
Foundational Mathematics Concepts for the High School to College Transition

Day 2 – July 10, 2012

Lecture Notes
Functions via formulas: Theorems of Pappus and more

- Meta comments
- Four questions handout:
  - The four questions:
    * How do you think about functions?
    * How do your students think about functions?
    * How would you like your students to think about functions?
    * How do we move them from (b) to (c)?
  - Reveal questions one at a time. Discuss with group.
  - Put brief answers on poster paper?
  - Summarize?
- Functions as formulas:
  - Already used straight lines in context – didn't focus so much on form (point slope vs. slope-intercept controversy).
    * Aside: Why are linear functions so important?
  - Polynomials–typically the next most complicated
    * The "form" controversy: a sum of monomials (powers) vs. a product of linear or quadratic terms (why both?)
    * Advantages of each form. Context dependent.
      · As a sum of monomials?
      · As a product of linear or quadratic terms?
    * Can we go from one to the other?
    * Aside: Why are polynomials so important?
- Areas and Volumes:
  - Area is . . . naturally quadratic (units are length squared)
  - Volume is . . . naturally cubic (units are length cubed)

- The theorems of Pappus:
  - Pappus of Alexandria: c. 290 - c. 350. One of last great Greek mathematicians, known for his "Collection," a compendium of mathematics in eight volumes. Volume VIII? contains "his" centroid theorems for volume and surface area of solids of revolution.
  - Aside: New math is old! Elimination goes back to Fertile Crescent in 2000 BC and China over 2000 yrs ago!
  - Starting point is the centroid or center of mass of a lamina (thin plate of constant density)
    * What is the centroid or center of mass?
    * A first example?
      · A see saw
      · $m_i x_i$ the *moment* of mass m_i with respect to the origin.
      · Balances at $\overline{x}$ if $m_1(x_1 - \overline{x}) = m_2(\overline{x} - x_2) \to \overline{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$.
    * Extrapolate the see-saw definition into hanging string definition.
    * For which lamina/shapes do we know the centroid easily and why?
    * Principle of symmetry.
  - Statement of theorems
    * The volume theorem: If lamina is rotated about an axis, the volume of the lamina is $V = 2\pi RA_L$, where R is the distance of the center of mass of the lamina to the axis of rotation.
    * The area the theorem: If a curve (possibly the boundary of lamina) is rotated about an axis, the surface area of surface of revolution is $A = 2\pi RL_C$, where R is the distance of the center of mass of the curve to the axis of rotation.

  Often these are set up so a coordinate axis is the axis of rotation.

- Volume Examples:
  - Volume of a solid cylinder: radius r and height h. Center of mass at $(h/2, r/2)$. Volume $= 2\pi(r/2)\cdot(rh) = \pi r^2 h$.
  - Volume of a solid torus: Bagel Cam! Volume $= 2\pi R \cdot \pi r^2 = 2\pi^2 R r^2$.
  - Variations:
    * Suppose the interior radius is always, say 2, then Volume $= 2\pi^2(2+r)r^2$. A polynomial in factored form. What is its graph?
    * Suppose the lamina is an ellipse! What is the center of mass? Suppose the semi-major axis is $a = 2r$ and the semi-minor axis is $b = r$. The area is $\pi ab = 2\pi r^2$. Replace in the previous formula. What does this say about the volume?

- Area examples: Do the above.

- Locating the center of mass using a pin, string & weight
  - An example. Volume of the outer half of a bagel. How do we find the center of mass?
  - Using the string, we find the center of mass is located 1 7/8 in. from center of 4 1/4 in. radius disk. So if the radius is r, the coordinates are $15/8/17/4 = (15/34)r$ from the center! Approximate center of mass is 0.441 radii. The exact value is $4/(3\pi) \approx 0.4244$. Off by 4%.

The Sabine Equation and rational functions

- An interesting example of rational functions of the form $A/(B+Cx)$. What do these look like?
- Developed by Walter Sabine (1868-1919), Harvard physics professor
- Idea:
  - Sounds reflects off/is absorbed by objects/materials in varying amounts/percentages
  - Increasing the absorbance of materials in a room decreases the time it takes for sounds to die out
  - This time can be calculated
  - The standard unit is the drop in 60 decibels. (Recall decibels are a logarithmic scale, so this is a drop of 10^6 in loudness).
  - This time is:
  $$\frac{0.049\text{Volume}}{(\text{Surface area}) \times \text{absorbance}}$$
  Different materials have acoustic absorbance, which is always a number between 0 and 1
  - A room 8 ft high by 10 by 14 ft. has plaster walls and ceiling. These have $a = 0.29$. Then the total reverberation times is:
  $$\frac{.049 * 8 * 10 * 14}{0.29 \cdot (140 + 2 * 8 * 24) + 140a} = \frac{1120}{52.2 + a140}$$
  - a is a number between 0 and 1, so we can look at domain.
  - For a wood floor for example, $a = 0.15$. So we get 0.749.
- Exercise: Design a problem with one or more variable quantities about the reverberation time of a room that you might use to explore rational functions.

2011 Summer Workshop, College of the Holy Cross
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The Group Activity: Four questions about functions June 28, 2011

- **Question 1:**

- **Question 2:**

- **Question 3:**

- **Question 4:**