DETERMINANTS ASSOCIATED TO ZETA MATRICES OF POSETS

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Abstract. We consider the matrix $\mathfrak{Z}_P = Z_P + Z_P^t$, where the entries of Z_P are the values of the zeta function of the finite poset P. We give a combinatorial interpretation of the determinant of $\mathfrak{Z}_P$ and establish a recursive formula for this determinant in the case in which P is a boolean algebra.

§1. Introduction

The theory of partially ordered sets (posets) plays an important role in enumerative combinatorics and beyond. For example, the Möbius inversion formula for posets generalizes several fundamental theorems including the number-theoretic Möbius inversion theorem. For a detailed review of posets and Möbius inversion we refer the reader to [S1], chapter 3, and [Sa]. Below we provide a short exposition of the basic facts on the subject following [S1].

A partially ordered set (*poset*) P is a set which, by abuse of notation, we also call P together with a binary relation, called a *partial order* and denoted $\leq$, satisfying:

(1) $x \leq x$ for all $x \in P$ (reflexivity).
(2) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).
(3) If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

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Two elements x and y are *comparable* if $x \leq y$ or $y \leq x$. Otherwise they are *incomparable.* We write $x < y$ to mean $x \leq y$ and $x \neq y$.

Definition. Let $n \in \mathbb{N}$. Consider the poset P_n of subsets of $[n]$ under the inclusion relation. This poset is called a *boolean algebra* of rank n. In [S1] it is denoted by $2^{[n]}$.

The *zeta function* ζ of a poset P is defined by $\zeta(x, y) = 1$ for all $x \leq y$ in P. The zeta function belongs to the incidence algebra $I(P)$ of P [S1]. If P is a locally finite poset (i.e. every interval in P is finite), the zeta function ζ is invertible in the algebra $I(P)$. Its inverse is called the *Möbius function* of P and is denoted by μ. Note that one can define μ inductively by

$$\mu(x, x) = 1, \text{ for all } x \in P,$$

$$\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z), \text{ for all } x < y \text{ in } P.$$

For the remainder of the article, P will be a poset with n elements and the partial order denoted by $\leq$. We choose a labelling $x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j \Longrightarrow i < j$.

Definition. The *zeta matrix* Z_P of a poset P is defined as the $n \times n$ matrix with entries

$$(Z_P)_{ij} = \begin{cases} 1 & \text{if } x_i \leq x_j \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \zeta(x_i, x_j) & \text{if } x_i \leq x_j \\ 0 & \text{otherwise} \end{cases}$$

Observe that, with the chosen labelling, the zeta matrix is unipotent upper triangular. Its non-zero entries are the values of the zeta function.

We define the matrix $\mathfrak{Z}_P$ by $\mathfrak{Z}_P = Z_P + Z_P^t$. In Section §2 we give a combinatorial interpretation of the determinant of $\mathfrak{Z}_P$. The main theorem of the paper evaluates the determinant of $\mathfrak{Z}_n := \mathfrak{Z}_{P_n}$ when P_n is the boolean algebra of rank n. More specifically, in Section §3, we prove the following recursive formula on n.

Main Theorem. *If $n \geq 3$ is odd, then* $\det(\mathfrak{Z}_n) = 0$*. If n is even, then*

$$\det(\mathfrak{Z}_n) = 2^{\alpha_n},$$

where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$.

Consider also the matrix $\mathfrak{M}_P$ defined by $\mathfrak{M}_P = M_P + M_P^t$, where $M_P = Z_P^{-1}$. The non-zero entries of M_P are the values of the Möbius function. We refer to M_P as the *Möbius matrix* of the poset P. We have the following theorem.

Theorem 1. $\det(\mathfrak{M}_P) = \det(\mathfrak{Z}_P)$.

PROOF. The theorem is a direct consequence of the following lemma. $\square$

Lemma 1. *Let U be an $n \times n$ matrix such that $\det(U) = 1$ and let $V = U^{-1}$. Then, $\det(U + U^t) = \det(V + V^t)$.*

PROOF. We have

$$V^t + V = (U^{-1})^t + U^{-1} = (U^{-1})^t U U^{-1} + (U^t)^{-1} U^t U^{-1} = (U^{-1})^t [U + U^t] U^{-1}.$$

Thus $\det(V + V^t) = \det((U^{-1})^t) \det(U + U^t) \det(U^{-1})$ and since $\det(U) = 1$, we have $\det(V + V^t) = \det(U + U^t)$. $\square$

The first author would like to thank Steve Fisk for suggesting the recursion of the Main Theorem. The Theorem was conjectured by Steve Fisk in an unpublished manuscript on orthogonal polynomials on posets.

§2. Combinatorial interpretation of $\det(\mathfrak{Z}_P)$

In this section, we discuss a combinatorial interpretation of $\det(\mathfrak{Z}_P)$, given in terms of the adjacency matrices of comparability graphs. Specifically, consider a poset P as in the previous section with $|P| = n$. The matrix $Y_P = Z_P - I_n$, in which the diagonal entries of Z_P are replaced by 0, can be interpreted as the adjacency matrix of a *directed graph* (digraph) G_P associated to the strict order relation $x < y$ in P. The vertices of G_P are the elements of P, and there is a directed edge from x to y if and only if $x < y$. Elsewhere in the literature G_P is called the *comparability graph* of the poset P. Similarly, the matrix $\mathfrak{Y}_P = Y_P + Y_P^t$ is the adjacency matrix of the directed graph D_P in which there are edges in *both* directions between each pair of distinct comparable elements $x, y \in P$. Then we have

$$\mathfrak{Z}_P = Z_P + Z_P^t = Y_P + Y_P^t + 2I_n = \mathfrak{Y}_P + 2I_n,$$

and $\det(\mathfrak{Z}_P) = \det(\mathfrak{Y}_P + 2I_n) = \chi(-2)$ where $\chi(t)$ is the characteristic polynomial $\chi(t) = \det(\mathfrak{Y}_P - tI_n)$ of the matrix $\mathfrak{Y}_P$. We will also call this the characteristic polynomial of the graph D_P. The coefficients of this characteristic polynomial are explicitly related to the number of collections of disjoint directed cycles in the graph D_P. For further details we refer the interested reader to [C].

§3. The Boolean algebra case

Let $[n] = \{1, 2, \dots, n\}$ and consider the poset $P_n = 2^{[n]}$ of subsets of $[n]$ under the inclusion relation. Let $\mathfrak{Z}_n$ be the matrix defined in §1. The main result of the paper is the following theorem.

Theorem 2. *If $n \geq 3$ is odd, then* $\det(\mathfrak{Z}_n) = 0$. *If n is even, then*

$$\det(\mathfrak{Z}_n) = 2^{\alpha_n},$$

where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$.

The proof of the theorem will rely on several lemmas. First, we identify a particularly useful labelling of $P_n = 2^{[n]}$ for our purposes, and we consider only this labelling for the remainder of the paper. Each subset $A \in P_n$ will be encoded as a binary vector $v(A)$ of length n:

$$v(A) = (v_n, v_{n-1}, \dots, v_1)$$

where

$$v_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if not.} \end{cases}$$

For example, in P_2, $v(\emptyset) = 00$, $v(\{1\}) = 01$, $v(\{2\}) = 10$ and $v(\{1,2\}) = 11$.

Our labelling of P_n induces the usual numerical ordering when we interpret $v(A)$ as the binary expansion of an integer m, with $0 \leq m \leq 2^n - 1$.

Using this labelling yields an interesting recursive structure in the matrices $\mathfrak{Z}_n$.

Lemma 2. *The matrices Z_n and $\mathfrak{Z}_n$ have the following properties.*

(1) *The entries of Z_n above the diagonal are the first 2^n rows in the Pascal triangle modulo 2.*

(2) *For $n \geq 2$, Z_n and $\mathfrak{Z}_n$ have block decompositions:*

$$Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix} \quad \text{and} \quad \mathfrak{Z}_n = \begin{pmatrix} \mathfrak{Z}_{n-1} & Z_{n-1} \\ Z_{n-1}^t & \mathfrak{Z}_{n-1} \end{pmatrix}.$$

(This statement also holds with $n = 1$ if we take $\mathfrak{Z}_0 = 2$, $Z_0 = 1$.)

(3) *The Z_n matrix sequence can be generated by a recursive procedure as follows. Given Z_{n-1}, to form Z_n we replace each entry 1 by a 2×2 block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and each entry 0 by a 2×2 zero matrix. From part (2), we get a similar recursive procedure for the $\mathfrak{Z}_n$ sequence.*

PROOF. These properties follow directly from the definition of the matrices Z_n and $\mathfrak{Z}_n$ and the properties of the preferred labelling on P_n. $\square$

To evaluate the determinant $\det(\mathfrak{Z}_n)$, we follow the general advice of [K] and *introduce parameters* in the matrix entries. Specifically, we consider the matrix:

$$\mathfrak{Z}_n(x,y) = xZ_n + yZ_n^t = \begin{pmatrix} xZ_{n-1} + yZ_{n-1}^t & xZ_{n-1} \\ yZ_{n-1}^t & xZ_{n-1} + yZ_{n-1}^t \end{pmatrix}$$

Using the Maple computer algebra system, the determinants of the first few of these matrices are found to be:

$$
\begin{array}{cc}
n & \det(\mathfrak{Z}_n(x,y)) \\
 & \\
1 & x^2+xy+y^2 \\
2 & (x+y)^2(x^2-xy+y^2) \\
3 & (x-y)^2(x^2+xy+y^2)^3 \\
4 & (x+y)^6(x^2-xy+y^2)^5 \\
5 & (x-y)^{10}(x^2+xy+y^2)^{11}
\end{array}
\tag{4}
$$

An interesting recurrence explains the patterns evident in these examples.

Lemma 3. *The determinants of the $\mathfrak{Z}_n(x,y)$ matrices are related by the following recurrence:*

$$\det(\mathfrak{Z}_{n+2}(x,y)) = (\det(\mathfrak{Z}_n(x,y))^2 \det(\mathfrak{Z}_{n+1}(x,-y)).$$

(Note that the right side involves both $\mathfrak{Z}_{n+1}$ and $\mathfrak{Z}_n$ and that y is negated in the second factor.)

PROOF. We use part (2) of Lemma 2 twice to form the following block decomposition of $\mathfrak{Z}_{n+2}(x,y)$:

$$\mathfrak{Z}_{n+2}(x,y) = \begin{pmatrix} xZ_n + yZ_n^t & xZ_n & xZ_n & xZ_n \\ yZ_n^t & xZ_n + yZ_n^t & 0 & xZ_n \\ yZ_n^t & 0 & xZ_n + yZ_n^t & xZ_n \\ yZ_n^t & yZ_n^t & yZ_n^t & xZ_n + yZ_n^t \end{pmatrix}$$

(where each entry is a block of size $2^n \times 2^n$). To evaluate the determinant, we perform block-wise row and column operations. To simplify the notation, we write $Z = Z_n$ First subtract row 4 from each of the first three rows to obtain:

$$\begin{pmatrix} xZ & xZ - yZ^t & xZ - yZ^t & -yZ^t \\ 0 & xZ & -yZ^t & -yZ^t \\ 0 & -yZ^t & xZ & -yZ^t \\ yZ^t & yZ^t & yZ^t & xZ + yZ^t \end{pmatrix}.$$

Subtract column 1 from each of the columns 2,3,4 to obtain:

$$\begin{pmatrix} xZ & -yZ^t & -yZ^t & -xZ - yZ^t \\ 0 & xZ & -yZ^t & -yZ^t \\ 0 & -yZ^t & xZ & -yZ^t \\ yZ^t & 0 & 0 & xZ \end{pmatrix}.$$

Next, subtract column 2 from column 4:

$$\begin{pmatrix} xZ & -yZ^t & -yZ^t & -xZ \\ 0 & xZ & -yZ^t & -xZ - yZ^t \\ 0 & -yZ^t & xZ & 0 \\ yZ^t & 0 & 0 & xZ \end{pmatrix}.$$

Add column 1 to column 4, then add row 4 in the resulting matrix to row 2:

$$\mathfrak{Z}' = \begin{pmatrix} xZ & -yZ^t & -yZ^t & 0 \\ yZ^t & xZ & -yZ^t & 0 \\ 0 & -yZ^t & xZ & 0 \\ yZ^t & 0 & 0 & xZ + yZ^t \end{pmatrix}.$$

Expanding along the last column we obtain

$$\det(\mathfrak{Z}_{n+2}(x,y)) = \det(\mathfrak{Z}') = \det(xZ + yZ^t) \det \begin{pmatrix} xZ & -yZ^t & -yZ^t \\ yZ^t & xZ & -yZ^t \\ 0 & -yZ^t & xZ \end{pmatrix}. \tag{5}$$

The first factor on the right of (5) is $\det(\mathfrak{Z}_n(x,y))$. Continuing with the 3×3 matrix, subtract column 3 from column 2:

$$\begin{pmatrix} xZ & 0 & -yZ^t \\ yZ^t & xZ + yZ^t & -yZ^t \\ 0 & -xZ - yZ^t & xZ \end{pmatrix},$$

then add row 3 to row 2:

$$\begin{pmatrix} xZ & 0 & -yZ^t \\ yZ^t & 0 & xZ - yZ^t \\ 0 & -xZ - yZ^t & xZ \end{pmatrix}.$$

Expanding along column 2, we have

$$\begin{aligned} \det(\mathfrak{Z}_{n+2}(x,y)) &= \det(\mathfrak{Z}_n(x,y))^2 \det \begin{pmatrix} xZ & -yZ^t \\ yZ^t & xZ - yZ^t \end{pmatrix} \\ &= \det(\mathfrak{Z}_n(x,y))^2 \det \begin{pmatrix} xZ - yZ^t & -yZ^t \\ xZ & xZ - yZ^t \end{pmatrix} \\ &= \det(\mathfrak{Z}_n(x,y))^2 \det(\mathfrak{Z}_{n+1}(x,-y)), \end{aligned}$$

as claimed. For the last equality, we perform row and column interchanges to put the final matrix shown into the form:

$$\begin{pmatrix} xZ - yZ^t & xZ \\ -yZ^t & xZ - yZ^t \end{pmatrix}$$

required for $\mathfrak{Z}_{n+1}(x,-y)$. $\square$

From the initial cases computed with Maple in (4) and the recurrence from Lemma 3, we see that there are nonnegative integers α_n, β_n such that

$$\det(\mathfrak{Z}_n(x,y)) = (x + (-1)^n y)^{\alpha_n} (x^2 - (-1)^n xy + y^2)^{\beta_n}. \tag{6}$$

Moreover, the recurrence from Lemma 3 implies that

$$\begin{cases} \alpha_{n+2} = 2\alpha_n + \alpha_{n+1}, \\ \beta_{n+2} = 2\beta_n + \beta_{n+1}. \end{cases} \tag{7}$$

We also have the following fact that is evident from (4):

Lemma 4. *For all* $n \geq 1$, $\alpha_n = \beta_n + (-1)^{n+1}$.

PROOF. This follows by induction. The base cases come from the Maple computations in (4) above: $\alpha_1 = 0, \beta_1 = 1$, and $\alpha_2 = 2, \beta_2 = 1$. For the induction step, assume that the claim of the lemma has been proved for all $\ell \leq k+1$. Then subtracting the two recurrences from (7) shows that

$$\alpha_{k+2} - \beta_{k+2} = 2(\alpha_k - \beta_k) + \alpha_{k+1} - \beta_{k+1} = 2(-1)^{k+1} + (-1)^{k+2} = (-1)^{k+1} = (-1)^{k+3}. \quad \square$$

Proof of Theorem 2: To determine the determinant of the original $\mathfrak{Z}_n = \mathfrak{Z}_n(1,1)$, we simply substitute $x = y = 1$ in (6). The factors of $x - y$ show immediately that $\det(\mathfrak{Z}_n) = 0$ if n is odd. Moreover, when n is even we have $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$. We solve the first recurrence in (7) for α_n by the standard method for second order linear recurrences with constant coefficients. The characteristic equation is $r^2 - r - 2 = 0$, whose roots are $r = 2, -1$. Hence $\alpha_n = c_1(2)^n + c_2(-1)^n$ for some constants c_1, c_2. The initial conditions $\alpha_1 = 0, \alpha_2 = 2$ show that $c_1 = 1/3, c_2 = 2/3$. Hence:

$$\alpha_n = \frac{1}{3}(2)^n + \frac{2}{3}(-1)^n.$$

Hence if n, and therefore also $n+2$, are even, we have

$$\alpha_{n+2} = \frac{1}{3}(2)^{n+2} + \frac{2}{3} = 4\left(\frac{1}{3}(2)^n + \frac{2}{3}\right) - 2 = 4\alpha_n - 2. \quad \square$$

Corollary 1. *If* n *is even, then*

$$\det(\mathfrak{Z}_n) = 2^{\frac{2^n+2}{3}}.$$

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