A CLASSICAL CHARACTERIZATION OF NEWFORMS WITH EQUIVALENT EIGENFORMS IN $S_{k+1/2}(4N,\chi)$

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ABSTRACT. We continue our investigation, begun in [8], of the Hecke structure of spaces of half-integral weight cusp forms $S_{k+1/2}(4N,\chi)$, where k and N are positive integers with N odd, and χ is an even quadratic Dirichlet character modulo $4N$. In the Hecke decomposition of these spaces, we determine contributions arising from newforms which are quadratic twists of newforms at lower levels. Combining this result with its counterpart in [8] regarding non-twists gives this paper's main result: necessary and sufficient conditions under which a given newform has equivalent cusp forms in $S_{k+1/2}(4N,\chi)$. Our result reformulates, in classical terms, the representation-theoretic conditions given by Flicker [7] and Waldspurger [32]. Our conditions involve easily-verified information about the primes dividing the level of the newform, and about the behavior of the newform under certain quadratic twists and Atkin-Lehner involutions. We apply our theorem to give explicit examples of twisted newforms having no equivalent half-integral weight cusp forms in any space $S_{k+1/2}(4N,\chi)$ as above.

1. INTRODUCTION

The study of simultaneous Hecke eigenforms, particularly newforms, is central to the theory of modular forms. Results in this paper concern the structure of spaces of half-integral weight cusp forms, as modules for the algebra generated by the Hecke operators. This *Hecke structure* is complex by comparison with the integral weight case: It is well-known that an integral-weight newform is determined up to constant multiple by its Hecke eigenvalues for almost all primes. As a consequence of this *strong multiplicity-one* result, we have the following decomposition of the space of cusp forms in terms of newforms:

$$S_{2k}(N) \cong \bigoplus_{d|N} \sigma(N/d) S^0_{2k}(d)$$

Date: March 16, 2003.
1991 *Mathematics Subject Classification.* 11F37, 11F32, 11F11.

where d runs over positive divisors of N, and σ is the divisor function. By contrast, we cannot express the Hecke structure of all spaces of *half-integral weight* cusp forms as succinctly; this is due to the lack of an analogous multiplicity-one result on the full space of newforms. One alternative in this case is to refine the space of half-integral weight newforms, obtaining a subspace in which a newform *is* determined by its Hecke eigenvalues. In [14], Kohnen produced such a refinement when the odd part of the level is squarefree. In several papers including [30] and [31], Ueda considered newforms of arbitrary level. As a function of the level, he determined the subspace on which a strong multiplicity-one result holds, thus providing a robust theory of newforms of half-integral weight.

A second approach is to focus on the full space of newforms, determining the multiplicity of distinct Hecke eigenforms with the same eigenvalues for almost all primes. This approach was taken by Shemanske in [23], and Shemanske & Walling in [24], in the case where the odd part of the level is squarefree. A related approach was taken by Ueda in [28], in the context of the Kohnen subspace. Ueda used trace identities to decompose the Kohnen subspace in terms of a direct sum of spaces of integral weight newforms. These "decompositions" are isomorphisms between modules for the Hecke algebras and therefore preserve Hecke eigenforms. Thus the Hecke structure of the half-integral weight space is determined by "pulling back" information from the integral weight side of the isomorphism.

In [8], we proved decompositions for the full space of half-integral weight cusp forms under certain conditions, utilizing Ueda's methods. We focused on $S_{k+1/2}(4p^m, \chi)$ with p an odd prime, m a positive integer, and χ an even quadratic Dirichlet character modulo $4p^m$. Providing complete decompositions in the case of more general levels is computationally arduous. However, partial decompositions have led to this paper's main result: classical conditions which determine whether a given newform F has equivalent half-integral weight cusp forms in any space $S_{k+1/2}(4N, \chi)$ for which N is odd, and χ is even and quadratic. This theorem reformulates, in the setting of classical modular forms, well-known results of Flicker and Waldspurger. Flicker [7] uses the pth components of the automorphic representation associated to F to state a necessary and sufficient condition (H1) under which F will have equivalent half-integral weight cusp forms at *some* level N, although his result does not characterize this level. When a second condition (H2) is also satisfied, Waldspurger [32] gives a means of constructing the subspace of half-integral weight cusp forms at a given level N which are equivalent to F. Waldspurger's method is complex however, and his result does not determine all levels for which this space will be nonzero.

Our result determines these levels explicitly. In particular, it gives the *minimal level* at which F has an equivalent half-integral weight cusp form. This is achieved using easily-verified conditions on the primes dividing the level of F, and the behavior of F under certain quadratic twists and Atkin-Lehner involutions. In Corollary 5.1 of [8], we proved this result in the case where the newform F is *not the quadratic twist of any newform of lower level.* In this paper, we prove the remaining case, thereby completing the theorem. Using tables of Cremona [5], [6] we provide examples of nonzero newforms which have no equivalent forms in any space $S_{k+1/2}(4N, \chi)$, under the above restrictions on N and χ.

The Hecke theory behind this paper's main result has exciting implications regarding the nonvanishing of L-functions associated to quadratic twists of newforms: Suppose a nonzero newform F satisfies Flicker and Waldspurger's (H1) and (H2) conditions, and that a half-integral weight cusp form g is equivalent to F. As shown by Waldspurger [32], the Fourier coefficients of g are proportional to the *central values* of the L-functions $L(F_D, s)$ associated to quadratic twists of F by fundamental discriminants D. Knowing the minimal level of such a cusp form g allows us to determine a lower bound on the smallest D for which this central value is nonzero. Through connections with work of Ono & Skinner [19], we aim to show that a positive proportion of fundamental discriminants D with a specified small number of prime factors satisfy the condition that the central value $L(F_D, k)$ is nonzero. (Preliminary results have been promising). This will provide a link between these Hecke structure results and the work of many, including Bump, Friedberg & Hoffstein, [4], [9], Goldfeld [10], Hoffstein & Luo [12], James [13], Kohnen [15], and Kohnen & Zagier [16], who have made significant contributions to fully characterizing those D for which $L(F_D, k) \neq 0$.

2. Statement of Main Theorem

Before stating our main result (Theorem 2.1), we must first establish some notation and terminology. Let k and M be positive integers with M odd, and let $t \in \{0, 1\}$. Let $S_{2k}(2^t M)$ be the space of cusp forms of weight $2k$ and level $2^t M$, and $S^0_{2k}(2^t M)$ the subspace generated by the newforms. For each positive integer n relatively prime to $2^t M$, consider the Hecke operator $T_{2k}(n)$ acting on $S_{2k}(2^t M)$. For each positive integer Q with $(Q, 2^t M/Q) = 1$, let W_Q denote the Atkin-Lehner involution, and write W_q if Q is a power of a single prime q. Let R_χ denote the twisting operator with respect to the Dirichlet character χ, and write R_p if χ is the Legendre symbol modulo p. In the half-integral weight setting, let $S_{k+1/2}(4N, \chi)$ denote the space of cusp forms of weight $k + 1/2$, level $4N$ and Dirichlet character χ modulo $4N$. For each positive integer n with $(n, 2N) = 1$, consider the Hecke operator $\widetilde{T}_{k+1/2}(n^2)$ acting on $S_{k+1/2}(4N, \chi)$. Moreover, if $k = 1$, restrict to the subspace $V_{3/2}(4N, \chi) \subseteq S_{3/2}(4N, \chi)$ mapping to integral weight cusp forms under the Shimura correspondence. (For definitions and further details, see [17], [30], and [28].)

A newform $F \in S_{2k}(2N)$ is *equivalent* to a cusp form $f \in S_{k+1/2}(4N, \chi)$ if f and F are Hecke eigenforms whose corresponding eigenvalues are equal for almost all primes p. That is, $f \mid \widetilde{T}_{k+1/2}(p^2) = \lambda_p f$ and $F \mid T_{2k}(p) = \lambda_p F$ for almost all p, where $\lambda_p \in \mathbb{C}$. Let $S_{k+1/2}(4N, \chi, F)$ denote the subspace of $S_{k+1/2}(4N, \chi)$ consisting of all forms equivalent to

F. We have the following direct sum, taken over all newforms F of levels dividing $2N$:

$$S_{k+1/2}(4N,\chi)=\bigoplus_{F} S_{k+1/2}(4N,\chi,F)$$

We will need to characterize a newform $F\in S_{2k}^0(2^tM)$ according to two conditions. The first is whether F is the quadratic twist of some newform of lower level. (For example, the quadratic twist modulo p of a newform in $S_{2k}^0(1)$ or $S_{2k}^0(p)$ yields a newform in $S_{2k}^0(p^2)$, by Theorem 6 of [1].) We decompose the space $S_{2k}^0(2^tM)$ according to such twists: First, let $S_{2k}^n(2^tM)$ denote the subspace of $S_{2k}^0(2^tM)$ generated by all newforms which are *not* quadratic twists of newforms of lower levels. Now consider all subsets B consisting of primes q which divide M with $\mathrm{ord}_q(M)=2$. Let R_B denote the quadratic twist by *all* primes in B. For each prime $q\in B$, let $\nu_q\in\{0,1\}$, and put $\nu(B)=\{\nu_q : q\in B\}$. Then for each choice of B and $\nu(B)$, consider the level

$$M_{B,\nu(B)}=\prod_{p\mid M,\,p\notin B} p^{\mathrm{ord}_p(M)}\prod_{q\in B} q^{\nu_q}$$

If we put $S_{2k}^n(2^tM_{B,\nu(B)})\mid R_B=\{F\mid R_B : F\in S_{2k}^n(2^tM_{B,\nu(B)})\}$, then Theorem 6 of [1] implies that $S_{2k}^n(2^tM_{B,\nu(B)})\mid R_B$ is contained in $S_{2k}^0(2^tM)$. Note that if $B=\emptyset$, we simply obtain $S_{2k}^n(2^tM)$. Moreover, by Proposition (A8) of [30], we have the following direct sum taken over all choices of B and $\nu(B)$:

$$S_{2k}^0(2^tM)=\bigoplus_{B,\nu(B)} S_{2k}^n(2^tM_{B,\nu(B)})\mid R_B \tag{1}$$

For example, $S_{2k}^0(p^2)=S_{2k}^n(p^2)\oplus S_{2k}^0(p)\mid R_p\oplus S_{2k}^0(1)\mid R_p$. We will use $S_{2k}^{n\perp}(2^tM)$ to denote the direct sum of all terms except the $B=\emptyset$ term. (For further details see [8], Section 2.2.)

Secondly, we characterize F by certain eigenvalues for the Atkin-Lehner involutions. If $F\in S_{2k}^n(2^tM)$, we consider the eigenvalues of F and of its twist $F\mid R_p$ for each prime p dividing M with $\mathrm{ord}_p(M)\geq 2$. For such a prime, let $\alpha_p,\beta_p\in\{1,-1\}$ and define the following subspaces of $S_{2k}^{n}(2^tM)$:

$$S_{2k}^{p\alpha_p\beta_p}(2^tM) = \{F \in S_{2k}^{n}(2^tM) : F \mid W_p = \alpha_p F \text{ and } F \mid R_p \mid W_p = \beta_p F \mid R_p\}$$

(These subspaces first appeared in [22], denoted S_I, S_{II}, S_{II_ψ}, and S_{III}.) One shows that

$$S_{2k}^{n}(2^tM) = \bigoplus_{\alpha_p,\beta_p \in \{1,-1\}} S_{2k}^{p\alpha_p\beta_p}(2^tM)$$

On the other hand, if $F \in S_{2k}^{n\perp}(2^tM)$, then $F = G \mid R_B$ for some choice of B and some newform $G \in S_{2k}^{n}(M_{B,\nu(B)})$. In this case, we decompose $S_{2k}^{n}(M_{B,\nu(B)})$ similarly with respect to any prime p dividing $M_{B,\nu(B)}$ with order at least two, and instead characterize F by the Atkin-Lehner eigenvalues of its related newforms G and $G \mid R_p$.

We now state our main theorem giving classical conditions under which a newform F in $S_{2k}^{0}(2^tM)$ has equivalent cusp forms in $S_{k+1/2}(4N,\chi)$:

Theorem 2.1. *Let $F \in S_{2k}^{0}(2^tM)$ with M odd and $t \in \{0,1\}$, and let $S_{2k}^{n}(2^tM_{B,\nu(B)}) \mid R_B$ and $S_{k+1/2}(4N,\chi,F)$ be as given above. Then $S_{k+1/2}(4N,\chi,F) = \{0\}$ for all odd positive integers N, and all even quadratic Dirichlet characters χ modulo $4N$, if and only if the following conditions hold:*

In the case $F \in S_{2k}^{n}(2^tM)$:

At least one prime p dividing M has $ord_p(M)$ even, and for any such prime p,

$$\text{either } p \equiv 1 \pmod 4 \text{ and } F \in S_{2k}^{p--}(2^tM)$$
$$\text{or } p \equiv 3 \pmod 4 \text{ and } F \in S_{2k}^{p-+}(2^tM)$$

In the case $F \in S_{2k}^{n}(2^tM_{B,\nu(B)}) \mid R_B$ for some nonempty B:

(1) *$\nu_q = 0$ for at least one $q \in B$ satisfying $q \equiv 3 \pmod 4$, OR*

(2) *At least one prime p dividing $M_{B,\nu(B)}$ has $ord_p(M_{B,\nu(B)})$ even, and for any such prime p,*

$$\text{either } p \equiv 1 \pmod 4 \text{ and } F \in S_{2k}^{p--}(2^t M_{B,\nu(B)}) \mid R_B$$
$$\text{or } p \equiv 3 \pmod 4 \text{ and } F \in S_{2k}^{p-+}(2^t M_{B,\nu(B)}) \mid R_B$$

In the event that $S_{k+1/2}(4N,\chi,F) \neq \{0\}$, the minimal level for this occurs when $N = M$.

Remark. We established the conditions in the case $F \in S_{2k}^{n}(2^t M)$ in Corollary 5.1 of [8]. The conditions in the case $F \in S_{2k}^{n}(2^t M_{B,\nu(B)}) \mid R_B$ are new and follow from the Hecke structure result given below in Theorem 4.1. (Corollary 4.4 part (1) establishes that these are the only possible cases.) □

3. Preliminaries

Theorem 2.1 relies upon the ability to decompose a space of cusp forms $S_{k+1/2}(4N,\chi)$ of given level and character in terms of its Hecke eigenforms. These *decompositions* take the form of isomorphisms between $S_{k+1/2}(4N,\chi)$ and direct sums of spaces of integral weight newforms. They are obtained using *trace identities*, as described below:

Let $tr(T \mid V)$ denote the trace of an operator T on a vector space V. If we have subspaces $S_{half} \subseteq S_{k+1/2}(4N,\chi)$ and $S_{whole} \subseteq S_{2k}(2N)$, then as described in [11], it can be shown that

$$tr(\widetilde{T}_{k+1/2}(n^2) \mid S_{half}) = tr(T_{2k}(n) \mid S_{whole}) \text{ for all } n \text{ with } (n,2N)=1$$
$$\iff S_{half} \cong S_{whole} \text{ as modules for the algebra generated by all the Hecke operators}$$

By explicitly calculating $tr\big(\widetilde{T}_{k+1/2}(n^2) \mid S_{k+1/2}(4N,\chi)\big)$ in the case where $(n,2N)=1$ and χ is even and quadratic, Ueda [28] established the following identities:

Theorem 3.1. *(Ueda [28]) Let N be a positive integer such that $2 \leq ord_2(N) = \mu \leq 4$ and put $M = 2^{-\mu}N$. Let χ be an even quadratic Dirichlet character modulo N and suppose that* *the conductor of χ is divisible by 8 if $\mu = 4$. Then for $k \geq 5$ and for all positive integers n with $(n, N) = 1$ we have the following relation:*

$$tr\big(\widetilde{T}_{k+1/2}(n^2) \mid S_{k+1/2}(N,\chi)\big) = tr\big(T_{2k}(n) \mid S_{2k}(N/2)\big) + \sum_{L_0} \Lambda(n, L_0) tr\big(W_{L_0} T_{2k}(n) \mid S_{2k}(2^{\mu-1} L_0 L_1)\big)$$

and for $k = 3$ we have the following relation:

$$tr\big(\widetilde{T}_{3/2}(n^2) \mid V_{3/2}(N,\chi)\big) = tr\big(T_2(n) \mid S_2(N/2)\big) + \sum_{L_0} \Lambda(n, L_0) tr\big(W_{L_0} T_2(n) \mid S_2(2^{\mu-1} L_0 L_1)\big)$$

where

(1) $\sum_{L_0}$ *runs over all square divisors L_0 of M with $L_0 > 1$,*

(2) *to each L_0 the corresponding L_1 is given by $L_1 = M \prod_{p|L_0} p^{-ord_p(M)}$, and*

(3) *the constant $\Lambda(n, L_0)$ is defined as follows:*

$$\Lambda(n, L_0) = \prod_{p|M} \lambda(p, n; ord_p(L_0)/2), \text{ with}$$

$$\lambda(p,n;a) = \begin{cases} 1 & \text{if } a = 0 \\ 1 + \left(\frac{-n}{p}\right) & \text{if } 1 \leq a \leq \left[\frac{ord_p(N)-1}{2}\right] \\ \chi_p(-n) & \text{if } ord_p(N) \text{ is even and } a = \frac{ord_p(N)}{2} \end{cases}$$

In [8], we used this identity to construct full decompositions for $S_{k+1/2}(4p^m, \chi)$, where p is an odd prime and $m \geq 0$. While we are able to provide full decompositions for other specified levels, the computations are arduous with several primes dividing the level. In proving Theorem 2.1 however, full decompositions are not needed. We require only the following chain of results:

(1) Given a newform $F \in S_{2k}^0(2^t M)$, Theorem 4.1 below tracks the "appearance" of F in the decomposition of $S_{k+1/2}(4\widehat{M}, \chi)$, where $\widehat{M}$ is any odd positive integer with the same prime factors as M, each occurring to odd exponent at least 3, and where

χ is any even quadratic Dirichlet character modulo $4\widehat{M}$. We proved this result for $F \in S_{2k}^{n}(2^t M)$ in Theorem 5.2 of [8]. The proof for $F \in S_{2k}^{n\perp}(2^t M)$ is given below.

(2) The dimension of $S_{k+1/2}(4\widehat{M},\chi,F)$ is given by the multiplicity with which F appears in the decomposition of $S_{k+1/2}(4\widehat{M},\chi)$, as shown in Corollary 4.4 below.

(3) Introducing additional prime factors to increase the level of the space of half-integral weight forms does not result in any additional appearances of the newform F. That is, if N is any odd positive integer divisible by $\widehat{M}$, then F appears in the decomposition of $S_{k+1/2}(4N,\chi)$ if and only if it appeared in the decomposition of $S_{k+1/2}(4\widehat{M},\chi)$. We proved this result for any $F \in S_{2k}^{0}(2^t M)$ in Theorem 5.6 & Corollary 5.7 of [8].

In proving the decomposition results in Theorem 4.1, we must eliminate all W-operators from Ueda's trace identity for $S_{k+1/2}(4\widehat{M},\chi)$. There are several necessary tools for doing this. We first require the commuting relationships given below:

Proposition 3.2. *([1], [2], [30]) Let $N, n,$ and k be positive integers, let ψ be a quadratic Dirichlet character of conductor f_ψ, and let Q be a positive divisor of N with $(Q, N/Q) = 1$. For any $F \in S_{2k}(N)$, the following hold:*

(1) *If $(n, Nf_\psi) = 1$, then $F \mid R_\psi \mid T_{2k}(n) = \psi(n) F \mid T_{2k}(n) \mid R_\psi$.*

(2) *If $(n, N) = 1$, then $F \mid T_{2k}(n) \mid W_Q = F \mid W_Q \mid T_{2k}(n)$.*

(3) *If $(Q, f_\psi) = 1$, then $F \mid R_\psi \mid W_Q = \psi(n) F \mid W_Q \mid R_\psi$.*

(4) *If Q' is another divisor of N such that $(Q', QN/Q') = 1$, then*
$F \mid W_{Q'} \mid W_Q = F \mid W_{Q'Q} = F \mid W_Q \mid W_{Q'}$.

Moreover, if $N = p^\nu M$, with p an odd prime not dividing M, and $\nu \in \{0,1\}$, then

(5) $F \mid R_p \mid W_{p^2} = \left(\frac{-1}{p}\right) F \mid R_p$.

We must also express the L_0-sums from the trace identity in terms of traces of W-operators on spaces of *newforms*. This is done using the following:

Proposition 3.3. *(Ueda [30]) Let A, B be finite sets consisting of primes such that $A \cap B = \emptyset$ and also let ℓ_p for $p \in A$, and m_q for $q \in B$ be any non-negative integers. Then for a positive integer n relatively prime to $\prod_{p\in A} p^{\ell_p} \prod_{q\in B} q^{m_q}$, we have the following identity:*

$$\begin{aligned} tr\big(W_B T_{2k}(n) \mid S_{2k}(\prod_{p\in A} p^{\ell_p} \prod_{q\in B} q^{m_q})\big) \\ = \sum_{\substack{(t_q)_{q\in B} \\ 0\le t_q \le [m_q/2]}} \sum_{\substack{(v_p)_{p\in A} \\ 0 \le v_p \le \ell_p}} \prod_{p\in A} (\ell_p + 1 - v_p) tr\big(W_B T_{2k}(n) \mid S^0_{2k}(\prod_{p\in A} p^{v_p} \prod_{q\in B} q^{m_q - 2t_q})\big) \end{aligned}$$

We must then decompose the spaces $S^0_{2k}(2^t M)$, isolating the subspace $S^n_{2k}(2^t M)$ closed under quadratic twists, and the subspaces $S^n_{2k}(M_{B,\nu(B)}) \mid R_B$ as defined above. We must further split $S^n_{2k}(2^t M)$ into eigenspaces depending on *each* odd prime p dividing M with $\mathrm{ord}_p(M) \geq 2$. Let Ω denote the set of all such primes. Then define

$$S_{2k}^{(p\alpha_p\beta_p)_{p\in\Omega}}(N) = \{F \in S^n_{2k}(N) : F \mid W_p = \alpha_p F, \text{ and } F \mid R_p \mid W_p = \beta_p F \mid R_p, \text{ for all } p \in \Omega\}$$

where $\alpha_p, \beta_p \in \{1, -1\}$ for each prime $p \in \Omega$. It is easy to see that

$$S^n_{2k}(N) = \bigoplus S_{2k}^{(p\alpha_p\beta_p)_{p\in\Omega}}(N)$$

where the direct sum is taken over all possible choices for the tuple $(p\alpha_p\beta_p)_{p\in\Omega}$.

We will have need of the following properties regarding the subspaces defined above:

Proposition 3.4. *([8], [30]) Let N be a positive integer with $ord_p(N) \geq 2$ for some odd prime p. The subspaces $S_{2k}^{p\alpha_p\beta_p}(N)$ as defined above behave under the action of the Hecke operators $T_{2k}(n)$, the involution W_p, and the twisting operator R_p in the following way:*

(1) *$S_{2k}^{p\alpha_p\beta_p}(N)$ is closed under the action of $T_{2k}(n)$ for $\alpha_p, \beta_p \in \{1, -1\}$ and $(n, N) = 1$.*
(2) *$S_{2k}^{p\alpha_p\beta_p}(N)$ is closed under the action of W_p for $\alpha_p, \beta_p \in \{1, -1\}$, .*

(3) *$S_{2k}^{p++}(N)$ and $S_{2k}^{p--}(N)$ are both closed under the action of R_p, while $S_{2k}^{p+-}(N) \mid R_p = S_{2k}^{p-+}(N)$ and $S_{2k}^{p-+}(N) \mid R_p = S_{2k}^{p+-}(N)$.*

Proposition 3.5. *([8]) Let the notation and terminology be as above. Then each summand $S_{2k}^{n}(M_{B,\nu(B)}) \mid R_B$ of $S_{2k}^{n\perp}(2^tM)$ is closed under the action of W_p for each odd prime $p \mid M$.*

Lastly, we will need to incorporate coefficients involving characters into our trace terms. This can be done according to the following result:

Lemma 3.6. *([11], [20], [8]) Let $k, N, M,$ and Q be positive integers and suppose that $(Q, f_\psi) = 1$, where f_ψ is the conductor of ψ, a primitive Dirichlet character modulo M. Let $N' = lcm(N, M^2)$. Then for all n satisfying $(n, N') = 1$:*

(1) $\psi(n) tr\big(T_{2k}(n) \mid S_{2k}^0(N)\big) = tr\big(T_{2k}(n) \mid S_{2k}^0(N) \mid R_\psi\big)$

(2) $\psi(n)\, tr\big(W_Q T_{2k}(n) \mid S_{2k}^0(N)\big) = tr\big(W_Q T_{2k}(n) \mid S_{2k}^0(N) \mid R_\psi\big)$.

Moreover, if ψ is quadratic, then

(3) $\psi(n)\, tr\big(W_Q T_{2k}(n) \mid S_{2k}^0(N) \mid R_\psi\big) = tr\big(W_Q T_{2k}(n) \mid S_{2k}^0(N)\big)$.

4. Hecke Structure Theorems

As discussed in Section 3, proving Theorem 2.1 requires three main steps. The first step is to track the "appearance" of newforms $F \in S_{2k}^0(2^tM)$ in the decomposition of a certain space of half-integral weight cusp forms, $S_{k+1/2}(4\widehat{M}, \chi)$. Let t and M be as above. For each prime p dividing M, put $b_p = \mathrm{ord}_p(M)$. Split the primes dividing M into the following three sets:

$$\mathcal{U} = \{p \mid M : b_p = 1\} \qquad \mathcal{E} = \{p \mid M : b_p \geq 2 \text{ is even}\} \qquad \mathcal{O} = \{p \mid M : b_p \geq 3 \text{ is odd}\}$$

Consider all subsets B as described in Section 3 (hence $B \subseteq \mathcal{E}$), and consider levels $\widehat{M} = \prod_{p \mid M} p^{a_p}$, where each a_p is an odd integer satisfying $a_p \geq \max\{3, \mathrm{ord}_p(M)\}$. Let χ be any even quadratic Dirichlet character modulo $4\widehat{M}$. Then we have the following

Theorem 4.1. *Let the notation and terminology be as above. For $k \geq 2$, the total contribution made by subspaces of $S_{2k}^0(2^t M)$ in the decomposition of $S_{k+1/2}(4\widehat{M}, \chi)$ is given by*

$$\bigoplus_{B \subseteq \mathcal{E}} \bigoplus_{\substack{q \in B \\ \nu_q = 0,1}} \bigoplus_{\substack{p \in \mathcal{E}-B \\ \alpha_p, \beta_p = \pm 1}} C(\mathcal{E}, B, \nu(B))\, S_{2k}^{(p\alpha_p\beta_p)_{p\in\mathcal{E}-B}}(2^t M_{B,\nu(B)}) \mid R_B$$

where the multiplicity $C(\mathcal{E}, B, \nu(B))$ is given by

$$\begin{aligned} C(\mathcal{E}, B, \nu(B)) = (2-t) \prod_{q \in B} \left[\frac{a_q}{2}\right] \prod_{p \in \mathcal{U}} (3\left[\frac{a_p}{2}\right] + 1) \prod_{p \in \mathcal{E}-B} (\left[\frac{a_p}{2}\right] + 1 - \frac{b_p}{2}) \\ \times \prod_{p \in \mathcal{O}} (a_p + 1 - b_p) \prod_{p \in \mathcal{E}-B} \left((2+\alpha_p) + \beta_p \left(\frac{-1}{p}\right)\right) \prod_{q \in B} (2 + (2-\nu_q)\left(\frac{-1}{q}\right)) \end{aligned}$$

For $k = 1$, the same result holds with $S_{k+1/2}(4\widehat{M}, \chi)$ replaced by $V_{3/2}(4\widehat{M}, \chi)$.

Remark. The multiplicity $C(\mathcal{E}, B, \nu(B))$ depends, in particular, on whether F is the quadratic twist of a newform of lower level, and on whether $\mathrm{ord}_p(M)$ is even for any primes p dividing M.

Also, in certain cases, this direct sum simplifies considerably. For instance, if $\mathcal{E} = \emptyset$, then all sums and several products are trivial, and we simply have the contribution

$$(2-t) \prod_{p \in \mathcal{U}} (3\left[\frac{a_p}{2}\right] + 1) \prod_{p \in \mathcal{O}} (a_p + 1 - b_p) S_{2k}^n(2^t M)$$

Even for $\mathcal{E} \neq \emptyset$, certain summands may simplify. For instance, in the term with $B = \mathcal{E}$, we do not split the space $S_{2k}^n(2^t M_{\mathcal{E},\nu(\mathcal{E})}) \mid R_{\mathcal{E}}$ into subspaces depending on primes, since $\mathcal{E} - B = \emptyset$. □

Proof. In Theorem 5.2 of [8], we previously established all contributions to the decomposition of $S_{k+1/2}(4\widehat{M}, \chi)$ which are made by the subspace $S_{2k}^n(2^t M)$. (These constitute the $B = \emptyset$ summands.) Proving Theorem 4.1 is therefore reduced to determining those contributions made by the subspaces $S_{2k}^n(2^tM_{B,\nu(B)}) \mid R_B$ for all nonempty B. This case is handled by techniques similar to those used in [8]. In Proposition 4.3 below, we first isolate and simplify all terms in the trace identity for $S_{k+1/2}(4\widehat{M},\chi)$ which contribute to $S_{2k}^n(2^tM_{B,\nu(B)}) \mid R_B$ for a fixed nonempty B. We then show by induction that these terms combine to give the desired structure of coefficients and subspaces.

Ueda's trace identity for $S_{k+1/2}(4\widehat{M},\chi)$ includes summands for each square divisor L_0 of $\widehat{M}$. Applying Proposition 3.3, each L_0-term is expressed as a sum of traces on spaces of newforms. In this sum, all primes dividing L_0 will occur with even exponent in the levels. Thus we may clearly disregard L_0-terms where any prime $p \in \mathcal{O}$ divides L_0. For convenience, we group the remaining summands by the set of prime divisors of L_0. For given subsets $P_{\mathcal{U}} \subseteq \mathcal{U}$ and $P_{\mathcal{E}} \subseteq \mathcal{E}$, we use "$P_{\mathcal{U}}P_{\mathcal{E}}$-sum" to refer to the sum of all L_0-terms for which the set of prime divisors of L_0 is $P_{\mathcal{U}} \cup P_{\mathcal{E}}$.

Proposition 4.3. *Let the notation and terminology be as above, and let $\mathcal{A}$ denote the full set of prime divisors of M. Suppose $B \neq \emptyset$. For any subsets $P_{\mathcal{U}} \subseteq \mathcal{U}$ and $P_{\mathcal{E}} \subseteq \mathcal{E}$, the $P_{\mathcal{U}}P_{\mathcal{E}}$-sum gives contributions to $S_{2k}^n(2^tM_{B,\nu(B)}) \mid R_B$ for certain values of ν_q. The total contribution to $S_{2k}^n(2^tM_{B,\nu(B)}) \mid R_B$ for all values of ν_q is derived from:*

$$\sum_{P_{\mathcal{E}}\subseteq\mathcal{E}} \sum_{\substack{q\in B\\ \nu_q=0,1}} K(P_{\mathcal{E}},B)\, tr\big(W_{P_{\mathcal{E}}\cap(\mathcal{E}-B)}T_{2k}(n) \mid S_{2k}^n(2^t \prod_{p\in\mathcal{A}-B} p^{b_p} \prod_{q\in B} q^{\nu_q}) \mid R_B\big)$$

where the coefficient $K(P_{\mathcal{E}},B)$ is given by

$$K(P_{\mathcal{E}},B) = (2-t) \prod_{q\in\mathcal{E}-P_{\mathcal{E}}} 2 \prod_{q\in B} \left[\frac{a_q}{2}\right] \prod_{p\in\mathcal{U}} \left(3\left[\frac{a_p}{2}\right]+1\right) \prod_{p\in\mathcal{O}} (a_p+1-b_p) \prod_{p\in\mathcal{E}-B} \left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)$$
$$\times \prod_{p\in P_{\mathcal{E}}\cap(\mathcal{E}-B)} \left(1+\left(\frac{-n}{p}\right)\right) \prod_{q\in P_{\mathcal{E}}\cap B} (2-\nu_q)\left(\frac{-1}{q}\right)$$

Proof. The $P_{\mathcal{U}}P_{\mathcal{E}}$-sum in Ueda's trace identity for $S_{k+1/2}(4\widehat{M},\chi)$ is given by

$$\prod_{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}}\left(1+\left(\frac{-n}{q}\right)\right)\sum_{\substack{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}\\ 1\le u_q\le\left[\frac{a_q}{2}\right]}} tr\big(W_{P_{\mathcal{U}}}W_{P_{\mathcal{E}}}T_{2k}(n)\mid S_{2k}(2\prod_{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}} q^{2u_q}\prod_{\substack{p\in\mathcal{A}\\ p\notin P_{\mathcal{U}}\cup P_{\mathcal{E}}}} p^{a_p})\big)$$

Applying Proposition 3.3 to convert to an expression involving spaces of newforms, and combining the nested sums that result, we obtain

$$\prod_{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}}\left(1+\left(\frac{-n}{q}\right)\right)\sum_{t=0}^{1}(2-t)\sum_{\substack{p\in\mathcal{A}-(P_{\mathcal{U}}\cup P_{\mathcal{E}})\\ 0\le v_p\le a_p}}\prod_{p}(a_p+1-v_p)$$
$$\times\sum_{Q\subseteq P_{\mathcal{U}}\cup P_{\mathcal{E}}}\sum_{\substack{q\in Q\\ 1\le t_q\le\left[\frac{a_q}{2}\right]}}\prod_{q}(\left[\frac{a_q}{2}\right]+1-t_q)\prod_{\substack{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}\\ q\notin Q}}\left[\frac{a_q}{2}\right] tr\big(W_QT_{2k}(n)\mid S_{2k}^0(2^t\prod_{\substack{p\in\mathcal{A}\\ p\notin P_{\mathcal{U}}\cup P_{\mathcal{E}}}} p^{v_p}\prod_{q\in Q} q^{2t_q})\big)$$

Not every Q-term gives contributions to $S_{2k}^n(2^tM_{B,\nu(B)})\mid R_B$. In fact, for a given $P_{\mathcal{U}}$ and $P_{\mathcal{E}}$, we must make the following restrictions on Q:

(1) $P_{\mathcal{U}}\subseteq Q$

(2) For all primes p in $\mathcal{E}-B$ (which may be empty), if $p\in P_{\mathcal{E}}$, then $p\in Q$.

Clearly if $q\in P_{\mathcal{U}}$, then q will appear in the level only if $q\in Q$ as well. Now instead suppose $p\in\mathcal{E}-B$. If $p\notin P_{\mathcal{E}}$, then p^{v_p} appears in the level and we need only choose the appropriate value for v_p. However if $p\in P_{\mathcal{E}}$, we must have $p\in Q$ in order to get a power of p in the level.

We keep only the terms for which Q satisfies both restrictions above. To represent this in the notation, put $Q=P_{\mathcal{U}}\cup Q_{\mathcal{E}}$, where $Q_{\mathcal{E}}\subseteq P_{\mathcal{E}}$. Rewriting the expression in terms of $Q_{\mathcal{E}}$ yields the following, in which the $Q_{\mathcal{E}}$ sum runs over all sets $Q_{\mathcal{E}}\subseteq P_{\mathcal{E}}$ which contain $(\mathcal{E}-B)\cap P_{\mathcal{E}}$:

$$\prod_{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}}\left(1+\left(\frac{-n}{q}\right)\right)\sum_{t=0}^{1}(2-t)\sum_{\substack{p\in\mathcal{A}-(P_{\mathcal{U}}\cup P_{\mathcal{E}})\\ 0\le v_p\le a_p}}\prod_{p}(a_p+1-v_p)\sum_{Q_{\mathcal{E}}}\sum_{\substack{q\in P_{\mathcal{U}}\cup Q_{\mathcal{E}}\\ 1\le t_q\le\left[\frac{a_q}{2}\right]}}\prod_{q}(\left[\frac{a_q}{2}\right]+1-t_q)\prod_{q\in P_{\mathcal{E}}-Q_{\mathcal{E}}}\left[\frac{a_q}{2}\right]$$
$$\times\, tr\big(W_QT_{2k}(n)\mid S_{2k}^0(2^t\prod_{\substack{p\in\mathcal{A}\\ p\notin P_{\mathcal{U}}\cup P_{\mathcal{E}}}} p^{v_p}\prod_{q\in P_{\mathcal{U}}\cup Q_{\mathcal{E}}} q^{2t_q})\big)$$

The following values for t_q and v_p achieve the desired space $S^n_{2k}(2^t M_{B,\nu(B)}) \mid R_B$:

(1) $v_p = b_p$ for all $p \in \mathcal{A} - (P_{\mathcal{U}} \cup P_{\mathcal{E}})$

(2) $t_q = 1$ for all $q \in P_{\mathcal{U}} \cup (B \cap Q_{\mathcal{E}})$

(3) $t_q = b_q/2$ for all $q \in (\mathcal{E} - B) \cap P_{\mathcal{E}}$.

(Note that our second restriction on $Q_{\mathcal{E}}$ implies that $(\mathcal{E} - B) \cap P_{\mathcal{E}} = (\mathcal{E} - B) \cap Q_{\mathcal{E}}$, hence we have accounted for all t_q.)

We keep only the term with v_p and t_q as chosen above, and separate out the primes occurring to exponent 2 in the level. (Such primes affect the content of the $S^{n\perp}$-space.) We eliminate the $W_{P_{\mathcal{U}}}$ operator using part (5) of Proposition 3.2. Following these computations, the remaining expression is:

$$(2-t) \prod_{q \in P_{\mathcal{U}} \cup P_{\mathcal{E}}} \left(1 + \left(\frac{-n}{q}\right)\right) \prod_{q \in P_{\mathcal{U}}} \left(\frac{-1}{q}\right) \prod_{\substack{p \in \mathcal{A} \\ p \notin \mathcal{E} \cup P_{\mathcal{U}}}} (a_p + 1 - b_p) \prod_{p \in \mathcal{E} - B} \left(\left[\frac{a_p}{2}\right] + 1 - \frac{b_p}{2}\right) \prod_{q \in \mathcal{E} - P_{\mathcal{E}}} 2 \prod_{q \in B \cup P_{\mathcal{U}}} \left[\frac{a_q}{2}\right]$$
$$\times \sum_{\substack{Q_{\mathcal{E}} \subseteq P_{\mathcal{E}} \text{ with} \\ ((\mathcal{E} - B) \cap P_{\mathcal{E}}) \subseteq Q_{\mathcal{E}}}} tr\Big(W_{Q_{\mathcal{E}}} T_{2k}(n) \mid S^0_{2k}\big(2^t \prod_{\substack{p \in \mathcal{A} \\ p \notin B \cup P_{\mathcal{U}}}} p^{b_p} \prod_{\substack{q \in B \cup P_{\mathcal{U}} \\ q \notin P_{\mathcal{E}} - Q_{\mathcal{E}}}} q^2\big)\Big)$$

For each $Q_{\mathcal{E}}$, the space of newforms in our trace term can be decomposed into a direct sum using equation (1) from Section 2. Summands include S^n and spaces of lower levels being twisted by primes in various subsets of $(B - (P_{\mathcal{E}} - Q_{\mathcal{E}})) \cup P_{\mathcal{U}}$. Since each summand is closed under the action of $W_{Q_{\mathcal{E}}}$ by Proposition 3.5, we may break up the trace expression across this sum. Not all of the resulting trace terms will yield a contribution for the desired space $S^n_{2k}(2^t M_{B,\nu(B)}) \mid R_B = S^n_{2k}(2^t \prod_{p \in \mathcal{A} - B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B$. We now determine the contributing terms:

If $P_{\mathcal{U}} = \emptyset$ and $P_{\mathcal{E}} - Q_{\mathcal{E}} = B$, the contributing term is the S^n subspace. In this case, the space of newforms in our trace term is $S^0_{2k}(2^t \prod_{p \in \mathcal{A} - B} p^{b_p})$, so the only possible summand to consider is S^n. In order to achieve the twist by R_B, we must employ the character coefficients in the trace term: The expansion of the coefficient $\prod_{q\in P_{\mathcal{E}}}\left(1+\left(\frac{-n}{q}\right)\right)$ contains the summand $\prod_{q\in B}\left(\frac{-n}{q}\right)$. Applying Lemma 3.6 part (1), the term with this summand in its coefficient will yield the space $S^0_{2k}(2^t\prod_{p\in\mathcal{A}-B}p^{b_p})\mid R_B$. Thus we obtain contributions to those spaces $S^n_{2k}(2^tM_{B,\nu(B)})\mid R_B$ for which all ν_q are zero.

If $P_{\mathcal{U}}\neq\emptyset$ or $P_{\mathcal{E}}-Q_{\mathcal{E}}\neq B$, clearly there are primes in $P_{\mathcal{U}}$ or B which appear in the level with exponent 2. We must therefore turn to $S^{n\perp}$ to reduce these levels as needed. In fact, it is easy to see that we require only the summands of $S^{n\perp}$ which involve twists by R_C, where $C=P_{\mathcal{U}}\cup(B-(P_{\mathcal{E}}-Q_{\mathcal{E}}))$. In order to achieve the twist by R_B instead of R_C, we again appeal to the character coefficients: The expansion of the coefficient $\prod_{q\in P_{\mathcal{U}}\cup P_{\mathcal{E}}}\left(1+\left(\frac{-n}{q}\right)\right)$ contains the summand $\prod_{q\in P_{\mathcal{U}}\cup(P_{\mathcal{E}}-Q_{\mathcal{E}})}\left(\frac{-n}{q}\right)$. We keep only the term with this summand in its coefficient. Applying Lemma 3.6 part (3) for each $q\in P_{\mathcal{U}}$, and part (2) for each $q\in P_{\mathcal{E}}-Q_{\mathcal{E}}$, we achieve the desired twist by R_B.

We further reduce the expression by eliminating certain W-operators. In particular, the $W_{Q_{\mathcal{E}}\cap B}$ piece of the operator can be eliminated by Proposition 3.2 part (5). The remaining piece $W_{Q_{\mathcal{E}}\cap(\mathcal{E}-B)}$ will be eliminated later using Proposition 3.4. We will obtain contributions corresponding to each piece of the coefficient $\prod_{q\in Q_{\mathcal{E}}\cap(\mathcal{E}-B)}\left(1+\left(\frac{-n}{q}\right)\right)$.

Following these computations as described, the terms which remain are:

$$\sum_{\substack{Q_{\mathcal{E}}\subseteq P_{\mathcal{E}}\text{ with}\\(\mathcal{E}-B)\cap P_{\mathcal{E}}\subseteq Q_{\mathcal{E}}}}\ \sum_{\substack{q\in B-(P_{\mathcal{E}}-Q_{\mathcal{E}})\\\nu_q=0,1}} L(Q_{\mathcal{E}},B)\prod_{q\in P_{\mathcal{E}}\cap B}\left(\frac{-1}{q}\right) tr\big(W_{Q_{\mathcal{E}}\cap(\mathcal{E}-B)}T_{2k}(n)\mid S^n_{2k}(2^t\prod_{p\in\mathcal{A}-B}p^{b_p}\prod_{\substack{q\in B\\ q\notin P_{\mathcal{E}}-Q_{\mathcal{E}}}}q^{\nu_q})\mid R_B\big)$$

where

$$L(Q_{\mathcal{E}},B)=(2-t)\prod_{p\in Q_{\mathcal{E}}\cap(\mathcal{E}-B)}\left(1+\left(\frac{-n}{p}\right)\right)\prod_{\substack{p\in\mathcal{A}\\ p\notin\mathcal{E}\cup P_{\mathcal{U}}}}(a_p+1-b_p)$$

$$\times\prod_{p\in\mathcal{E}-B}\left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)\prod_{q\in\mathcal{E}-P_{\mathcal{E}}}2\prod_{q\in B\cup P_{\mathcal{U}}}\left[\frac{a_q}{2}\right]$$

Notice that for $Q_{\mathcal{E}}$ as restricted, we have $Q_{\mathcal{E}} \cap (\mathcal{E} - B) = P_{\mathcal{E}} \cap (\mathcal{E} - B)$, so in fact the trace term can be written without explicit dependence on $Q_{\mathcal{E}}$. We then employ a simple combinatorial argument to simplify the sums: For $q \in B$, the number of choices of ν_q for which we get contributions to $S_{2k}^n(2^t \prod_{p \in \mathcal{A}-B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B$ will depend on the choice of $Q_{\mathcal{E}}$. In our sums over ν_q, primes in $B - P_{\mathcal{E}}$ will occur in the level for all choices of ν_q. Now consider a prime $q \in P_{\mathcal{E}}$. If $q \in Q_{\mathcal{E}}$, then q^{ν_q} appears in the expression for the level, so we get contributions for both $\nu_q = 0$ and 1. However, if $q \notin Q_{\mathcal{E}}$, then q^{ν_q} is absent from the expression, so we get contributions only for $\nu_q = 0$. Thus we have $(2 - \nu_q)$ contributions.

Applying this argument for all relevant primes reduces the expression to

$$\sum_{\substack{q \in B \\ \nu_q = 0,1}} \prod_{q \in P_{\mathcal{E}} \cap B} (2 - \nu_q) \Big(\frac{-1}{q} \Big) L(P_{\mathcal{E}}, B)\, tr\big(W_{P_{\mathcal{E}} \cap (\mathcal{E}-B)} T_{2k}(n) \mid S_{2k}^n(2^t \prod_{p \in \mathcal{A}-B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B \big)$$

where $L(P_{\mathcal{E}}, B)$ is obtained by replacing $Q_{\mathcal{E}}$ with $P_{\mathcal{E}}$ in the expression for $L(Q_{\mathcal{E}}, B)$.

Finally, we must sum these expressions over all subsets $P_{\mathcal{U}} \subseteq \mathcal{U}$. Rearranging the resulting sum to show the dependence on $\mathcal{U}$ and $P_{\mathcal{U}}$ reveals a factor of $\sum_{P_{\mathcal{U}} \subseteq \mathcal{U}} \prod_{p \in P_{\mathcal{U}}} \left[\frac{a_p}{2}\right] \prod_{p \in \mathcal{U} - P_{\mathcal{U}}} a_p$. A simple induction on $|\mathcal{U}|$ then shows that

$$\sum_{P_{\mathcal{U}} \subseteq \mathcal{U}} \prod_{p \in P_{\mathcal{U}}} \left[\frac{a_p}{2}\right] \prod_{p \in \mathcal{U} - P_{\mathcal{U}}} a_p = \prod_{p \in \mathcal{U}} \left(3 \left[\frac{a_p}{2}\right] + 1 \right)$$

Incorporating this result, we obtain the desired expression. This completes the proof of Proposition 4.3. □

The result from Proposition 4.3 includes all terms which will yield contributions to *some or all* of the spaces $S_{2k}^n(2^t M_{B,\nu(B)}) \mid R_B$, for a fixed set B. To complete the proof of Theorem 4.1, we must show that the expression reduces to give the structure of coefficients and subspaces as stated. This will require first eliminating the remaining W-operators and then evaluating the sum over $P_{\mathcal{E}}$. (Note that in the case $\mathcal{E} = B$, there are no remaining W-operators to address.)

For each prime $p \in P_{\mathcal{E}} \cap (\mathcal{E} - B)$, we incorporate the coefficient $\left(1 + \left(\frac{-n}{p}\right)\right)$ into the trace expression as follows: After writing S^n as the direct sum of $S_{2k}^{p\alpha_p\beta_p}$ for $\alpha_p, \beta_p = \pm 1$, we apply part (3) of Proposition 3.4, followed by part (1) of Lemma 3.6. We then evaluate the sum over $P_{\mathcal{E}}$ in our expression by using induction on $|\mathcal{E}|$ to show that

$$\sum_{P_{\mathcal{E}} \subseteq \mathcal{E}} \prod_{q \in \mathcal{E} - P_{\mathcal{E}}} 2 \sum_{\substack{p \in P_{\mathcal{E}} \cap (\mathcal{E}-B) \\ \alpha_p, \beta_p = \pm 1}} \prod_{p \in P_{\mathcal{E}} \cap (\mathcal{E}-B)} \left(\alpha_p + \beta_p \left(\frac{-1}{p}\right)\right)$$

$$\times \sum_{\substack{q \in B \\ \nu_q = 0,1}} \prod_{q \in P_{\mathcal{E}} \cap B} (2 - \nu_q) \left(\frac{-1}{q}\right) tr\left(T_{2k}(n) \mid S_{2k}^{(p\alpha_p\beta_p)_{p \in P_{\mathcal{E}} \cap (\mathcal{E}-B)}}(2^t \prod_{p \in \mathcal{A}-B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B\right)$$

$$= \sum_{\substack{q \in B \\ \nu_q = 0,1}} \sum_{\substack{p \in \mathcal{E}-B \\ \alpha_p, \beta_p = \pm 1}} \prod_{p \in \mathcal{E}-B} \left((2 + \alpha_p) + \beta_p \left(\frac{-1}{p}\right)\right)$$

$$\times \prod_{q \in B} \left(2 + (2 - \nu_q) \left(\frac{-1}{q}\right)\right) tr\left(T_{2k}(n) \mid S_{2k}^{(p\alpha_p\beta_p)_{p \in \mathcal{E}-B}}(2^t \prod_{p \in \mathcal{A}-B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B\right)$$

The base case of the induction clearly holds. Now suppose the equation holds for any set $\mathcal{E}$ containing at most ℓ primes. Then for $\mathcal{E}$ with $|\mathcal{E}| = \ell + 1$, separate off one prime q' which we may assume lies in B (since $B \neq \emptyset$). Now write $\mathcal{E} = \mathcal{E}' \cup \{q'\}$ and split the subsets $P_{\mathcal{E}} \subseteq \mathcal{E}$ into two types: (1) $P_{\mathcal{E}} = P_{\mathcal{E}'}$ (including $\emptyset$), and (2) $P_{\mathcal{E}} = P_{\mathcal{E}'} \cup \{q'\}$ for some $P_{\mathcal{E}'}$ as in (1). In addition, write $B = B' \cup \{q'\}$, so that we may rewrite the left-hand side of the above equation in terms of $P_{\mathcal{E}'}$, $\mathcal{E}'$, and B'. This results in the following:

$$\left(2 + (2 - \nu_{q'}) \left(\frac{-1}{q'}\right)\right) \Bigg[\sum_{P_{\mathcal{E}'} \subseteq \mathcal{E}'} \prod_{q \in \mathcal{E}' - P_{\mathcal{E}'}} 2 \sum_{p \in P_{\mathcal{E}'} \cap (\mathcal{E}'-B')} \left(\alpha_p + \beta_p \left(\frac{-1}{p}\right)\right) \sum_{\substack{q \in B \\ \nu_q = 0,1}} \prod_{q \in P_{\mathcal{E}'} \cap B'} (2 - \nu_q) \left(\frac{-1}{q}\right)$$

$$\times\, tr\left(T_{2k}(n) \mid S_{2k}^{(p\alpha_p\beta_p)_{p \in P_{\mathcal{E}'} \cap (\mathcal{E}'-B')}}(2^t \prod_{\mathcal{A}-B} p^{b_p} \prod_{q \in B} q^{\nu_q}) \mid R_B\right) \Bigg]$$

Now we apply the inductive hypothesis to the bracket and simplify, noting that $\mathcal{E}' - B' = \mathcal{E} - B$. Finally, summing the result over all choices for the subset B yields

$$\sum_{B \subseteq \mathcal{E}} \sum_{\substack{q \in B \\ \nu_q = 0,1}} \sum_{\substack{p \in \mathcal{E}-B \\ \alpha_p, \beta_p = \pm 1}} C(\mathcal{E}, B, \nu(B)) tr\left(T_{2k}(n) \mid S_{2k}^{(p\alpha_p\beta_p)_{p \in \mathcal{E}-B}}(2^t M_{B,\nu(B)}) \mid R_B\right)$$

Since this trace involves only Hecke operators, it yields the isomorphism as desired. This concludes the proof of Theorem 4.1. □

Our second step in proving Theorem 2.1 is to use the Hecke structure from Theorem 4.1 to determine the dimension of $S_{k+1/2}(4\widehat{M},\chi,F)$, the subspace of $S_{k+1/2}(4\widehat{M},\chi)$ generated by the eigenforms equivalent to our newform F. These dimensions are given in the following:

Corollary 4.4. *Let the notation and terminology be as in Theorem 4.1 and let $F \in S^0_{2k}(2^tM)$ be a newform. Then*

(1) *F is an element of $S^n_{2k}(2^tM_{B,\nu(B)}) \mid R_B$ for some $B \subseteq \mathcal{E}$ and some choice of $\nu(B)$, and*

(2) *The dimension of the space $S_{k+1/2}(4\widehat{M},\chi,F,)$ is given by*

$$dim(S_{k+1/2}(4\widehat{M},\chi,F,)) = (2-t)\prod_{q\in\mathcal{B}}\left[\frac{a_q}{2}\right]\prod_{p\in\mathcal{U}}(3\left[\frac{a_p}{2}\right]+1)\prod_{p\in\mathcal{E}-B}(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2})$$
$$\times\prod_{p\in\mathcal{O}}(a_p+1-b_p)\prod_{p\in\mathcal{E}-B}\left((2+\alpha_p)+\beta_p\left(\frac{-1}{p}\right)\right)\prod_{q\in B}(2+(2-\nu_q)\left(\frac{-1}{q}\right))$$

Remark. As in Theorem 4.1, the expression given in (2) is considerably simpler in certain cases. For instance, if $\mathcal{E}=\emptyset$, we do not split the space $S^0_{2k}(2^tM)$ into subspaces, and for every newform $F \in S^0_{2k}(2^tM)$ we simply have

$$\dim(S_{k+1/2}(4\widehat{M},\chi,F,)) = (2-t)\prod_{p\in\mathcal{U}}\left(3\left[\frac{a_p}{2}\right]+1\right)\prod_{p\in\mathcal{O}}(a_p+1-b_p)$$

Also, for us the significant factors in the expression for $\dim(S_{k+1/2}(4\widehat{M},\chi,F,))$ are those which may be zero. For instance, suppose we have $p\in\mathcal{E}$ satisfying $p\equiv 1 \pmod 4$, and $F\in S^{p--}_{2k}(2^tM)$. Then the coefficient has a factor of $1-\left(\frac{-1}{p}\right)$ which is zero in this case. Therefore $\dim(S_{k+1/2}(4\widehat{M},\chi,F,)) = 0$, so F has no equivalent half-integral weight Hecke eigenforms of level $4\widehat{M}$ and character χ. □

Proof. Clearly (2) follows immediately from Theorem 4.1, once (1) is established. The proof for the case $B = \emptyset$ was given previously in Corollary 5.5 in [8], however, the following argument encompasses all cases:

Let F be a newform in $S_{2k}^0(2^tM)$. If $\mathcal{E} = \emptyset$, $S_{2k}^0(2^tM)$ does not split into subspaces, and therefore (1) holds trivially. Otherwise, write

$$S_{2k}^0(2^tM) = \bigoplus_{\substack{p\in\mathcal{E}\\ \alpha_p,\beta_p=\pm1}} S_{2k}^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}(2^tM) \oplus \bigoplus_{\substack{q\in\mathcal{E}\\ \nu_q=0,1}} S_{2k}^n(2^tM_{\mathcal{E},\nu(\mathcal{E})}) \mid R_{\mathcal{E}}$$
$$\oplus \bigoplus_{\substack{B\subset\mathcal{E},\, q\in B\\ \nu_q=0,1}} \bigoplus_{\substack{p\in\mathcal{E}-B\\ \alpha_p,\beta_p=\pm1}} S_{2k}^{(p\alpha_p\beta_p)_{p\in\mathcal{E}-B}}(2^tM_{B,\nu(B)}) \mid R_B$$

Each of the summands is a space which is preserved by the action of the Hecke operators $T_{2k}(n)$ for all n relatively prime to $2M$: This follows from part (1) of Proposition 3.4, since $T_{2k}(n)$ and the twisting operators commute up to constant multiple (by part (1) of Proposition 3.2). Now $S_{2k}^0(2^tM)$ has a basis $\mathcal{B}$ consisting of newforms, and moreover, using standard facts from linear algebra, each of the summands has a basis consisting of some subset of $\mathcal{B}$. Since F is a newform, it must therefore be a constant multiple of some element of $\mathcal{B}$ (by multiplicity-one). Hence F must be an element of one of the summands. □

The final step in proving Theorem 2.1 is to show that the dimension of $S_{k+1/2}(4\widehat{M},\chi,F)$ in fact gives us the dimension of $S_{k+1/2}(4N,\chi,F)$ for any odd N and any even quadratic χ modulo $4N$. We proved this result for all newforms in $S_{2k}^0(2^tM)$ in Corollary 5.7 of [8], which we state here for completeness:

Proposition 4.6. *([8]) Let $F \in S_{2k}^0(2^tM)$ be a newform, with t and M as above. For $k \geq 2$ (resp. $k = 1$), if F does not appear in the decomposition of $S_{k+1/2}(4\widehat{M},\chi)$ (resp. $V_{3/2}(4\widehat{M},\chi)$) for any positive integer $\widehat{M} = \prod_{p|M} p^{\alpha_p}$ with odd integers $\alpha_p \geq 3$, and for any even quadratic Dirichlet character modulo $4\widehat{M}$, then F does not appear in the decomposition* *of* $S_{k+1/2}(4N,\chi')$ *(resp.* $V_{3/2}(4N,\chi')$*) for any odd positive integer* N *and any even quadratic Dirichlet character* χ' *modulo* $4N$.

Finally, combining this result with Corollary 4.4, one can determine whether a newform $F \in S_{2k}^0(2^t M)$ has equivalent half-integral weight cusp forms in a given space $S_{k+1/2}(4N,\chi)$. One need only check whether the coefficient corresponding to F in Corollary 4.4 is nonzero.

5. Examples

Theorem 2.1 indicates precisely which subspaces of newforms will be "missing" from the decompositions of $S_{k+1/2}(4N,\chi)$ for all N and χ as above. Consequently, any nonzero forms in these spaces will not be in the image of the Shimura lift from any such space $S_{k+1/2}(4N,\chi)$. We give examples of these "missing newforms," computed using Cremona's tables [5], [6] for rational newforms of weight 2. These tables list the following identifying information:

(1) The Hecke eigenvalues λ_p of F for $T_2(p)$ where p does not divide the level of F, and $p \leq 100$.
(2) The eigenvalues, either $+1$ or -1, of F for W_q where q divides the level of F and $q \leq 100$.

In [8], we used these tables to show that $\dim(S_2^{13--}(338)) \geq 2$ and $\dim(S_2^{19-+}(722)) \geq 1$. This illustrates nonzero newforms satisfying the set of conditions given in part (1) of Theorem 2.1. Here we will give similar examples for forms satisfying the remaining possibilities. First, we describe the process of utilizing Cremona's tables:

Suppose $F(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a normalized newform of weight 2 and level M. For a prime $p \mid M$, we then have $F \mid R_p = \sum_{n=1}^{\infty} b(n)e^{2\pi i n z}$ where $b(n) = \left(\frac{n}{p}\right)a(n)$. If $p \mid n$, $b(n) = 0$. Otherwise, corresponding $a(n)$ and $b(n)$ may differ only in sign, and will differ precisely when n is a quadratic non-residue modulo p. Since $a(n) = \lambda_n$, the eigenvalue of F

for the Hecke operator $T_2(n)$, we have a relationship between the Hecke eigenvalues of F and those of $F \mid R_p$. We can therefore use the eigenvalue information in the tables to determine whether $F \in S_2^n(M)$ or $F = G \mid R_B$ for some newform G at a lower level. We can then determine the piece of the corresponding S^n-space to which F or G belongs with respect to a given prime. (that is, whether $F \in S_2^{p++}(M)$, and so forth).

Example 5.1 Using this method, we have

$$\dim(S_2^{5--}(150) \mid R_3) \geq 2$$

We have $S_2^{5--}(150) \mid R_3 \subseteq S_2^0(450)$, and Cremona lists 7 distinct rational newforms of level 450, called $450A$ through $450G$. Examining the Hecke eigenvalues of these forms and of the rational newforms at level 150, we find that $450B, D, E$, and F are elements of $S_2^n(450)$ while $450A, C$, and G are twists by R_3 of $150B, A$, and C respectively. Moreover, we determine that $150A \mid R_5 = 150B$ and $150B \mid R_5 = 150A$, hence $150A$ and B are both in $S_2^n(150)$. Finally, checking the sign of the W_5-eigenvalues shows that $150A$ and B are in $S_2^{5--}(150)$, therefore $450A$ and C are in $S_2^{5--}(150) \mid R_3$. Thus $\dim(S_2^{5--}(150) \mid R_3) \geq 2$, giving an example of newforms satisfying the final set of conditions given in Theorem 2.1. □

Example 5.2 Similar calculations show that

$$\dim(S_2^{7-+}(490) \mid R_5) \geq 4$$

which also illustrates the third case of the theorem. □

Example 5.3 Finally, we compute that $19A \mid R_3 = 171B$, which shows that

$$\dim(S_2^n(19) \mid R_3) \geq 1$$

This illustrates at least one newform satisfying the second set of conditions of Theorem 2.1.

□

Corresponding to each part of Theorem 2.1, we have found additional examples of nonzero newforms satisfying the stated conditions. The ease with which these examples are computed suggests that there may in fact be many nonzero newforms which are not in the image of the Shimura lift from any $S_{k+1/2}(4N,\chi)$ with N and χ as above.

We also remark that while Cremona's tables deal exclusively with rational newforms, tables of Stein [27] extend this data to include eigenvalue information for *all* newforms of weight 2. Using these tables, any of our examples could be revisited to find the exact dimension of the subspace involved.

6. Conclusion

In this paper, we have used the explicit Hecke structure of spaces $S_{k+1/2}(4N,\chi)$ to provide easily-verified conditions under which a given newform will have equivalent forms in these spaces. We thereby illustrated the image of the Shimura lift from such spaces, and we provided examples which show that this lift need not be surjective. It would be interesting to know whether there are *always* missing subspaces of positive dimension. We are currently developing algorithms to compute the dimensions of the missing subspaces for any specified level. (These algorithms use MAGMA and are based on trace formulas of Ross [21], Yamauchi [33], and Saito-Yamauchi [22].)

It is possible to prove Hecke structure results for $S_{k+1/2}(N,\chi)$ for almost all levels N (with χ even and quadratic), using methods similar to those used in proving Theorem 4.1. This is due to the fact that trace identities handling most remaining cases of N are given by Ueda in [29]. We are currently adapting our techniques to handle these additional cases, with the goal of extending Theorem 2.1. The case of $S_{k+1/2}(64M,\chi)$ is of particular interest, as these decompositions also yield newforms which satisfy Waldspurger's (H2) condition. Proving an analog of Theorem 2.1 in this case will allow us to firmly establish the important connection between Hecke structure theory and the nonvanishing of central values of quadratic twists of L-functions, as described in the introduction.

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