Continua of Central Configurations with a Negative Mass in the *n*-Body Problem

John B. Little

Department of Mathematics and Computer Science
College of the Holy Cross, Worcester MA

AMS Special Session on Celestial Mechanics
JMM, San Diego

January 9, 2013

Work done at PURE Math 2012 REU-type program at University of Hawai'i at Hilo; joint with

- Julian Hachmeister (undergraduate, UH Hilo)
- Jasmine McGhee (undergraduate, Loyola Marymount)
- Roberto Pelayo (UH Hilo)
- Spencer Sasarita (undergraduate, U Arizona)

To appear, *Celestial Mechanics and Dynamical Astronomy*.

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- (Setting $G = 1$ and writing $\mathbf{q}_i$ for location of *i*th body):

$$\mathbf{A}_i = \sum_{j \neq i} \frac{m_j(\mathbf{q}_j - \mathbf{q}_i)}{r_{ij}^3} = \omega^2(\overline{\mathbf{q}} - \mathbf{q}_i)$$

The big problem

- A major question here is: Given n masses $m_1, \ldots, m_n$, at how many different locations can these be placed to get central configurations? (Usually in $\mathbb{R}^2$ or $\mathbb{R}^3$, but makes sense mathematically in higher dimensions too.)

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- More precise form of question: Is the set of equivalence classes of (planar, or ...) central configurations *finite*? On Smale's 21st century problem list.

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- Only fairly limited special cases known in general
- Question is subtle algebraically. For instance, by work of Gareth Roberts (Physica D 127 (1999), 141-145), there collections of $n = 5$ masses, one negative, for which there is a *curve* of equivalence classes of c.c.'s (a "continuum")

Geometry of Roberts' "rhombus +1"

- Choose coordinates so

$$\mathbf{q}_0 = (0,0), \mathbf{q}_1 = (\cos(t), 0) = -\mathbf{q}_2, \mathbf{q}_3 = (0, \sin(t)) = -\mathbf{q}_4.$$

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- The center of mass of the configuration, $\overline{\mathbf{q}}$, is located at the origin.

Roberts' "rhombus +1"

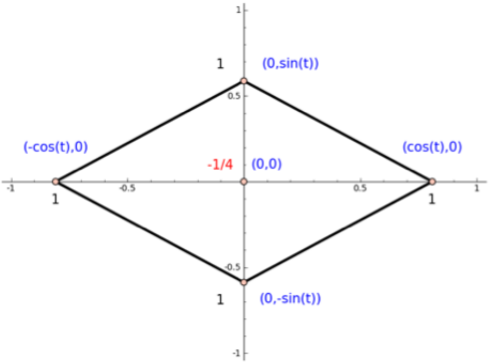

Figure: Rhombus with Roberts' parametrization

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- Therefore

$$\mathbf{A}_3 = (0, -2\sin(\theta)) = 2(0, -\sin(\theta)) = 2(\overline{\mathbf{q}} - \mathbf{q}_3).$$

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- $\overline{\mathbf{q}}$ is fixed at the origin and the distances from the 0th body are changing but the distances between consecutive vertices of the rhombus are not
- Therefore, we have found a continuum of inequivalent c.c.'s one for each θ in the interval $0 < \theta < \frac{\pi}{2}$.

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- They came up with a beautiful construction and a whole infinite family of additional examples, *but* only in $\mathbb{R}^{2k}$ for $k \geq 2$.
- The found their examples by looking at Roberts' construction in a different way (but can also make them look similar, and that's what we'll do in this talk)

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- $\mathbf{A}_0$, $\mathbf{A}_4$ in this sub-configuration also zero.
- Similarly for other sub-configuration $\{\mathbf{q}_1, \mathbf{q}_0, \mathbf{q}_2\}$.

Neutral configurations

Definition 1

We will say a configuration of $\ell > 1$ bodies is **neutral** if the gravitational acceleration on each body is zero.

Easy to see that neutral configurations are only possible if at least one mass is negative.

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- Place a new mass m_0 at the center of mass $\Rightarrow$ still central
- Can take $\mathbf{q}_0 = (0, 0)$ and

$$\mathbf{q}_j = \left(\cos\left(\frac{2\pi j}{n}\right), \sin\left(\frac{2\pi j}{n}\right)\right)$$

for $j = 1, \ldots, n$.

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  for $j = 1, \ldots, n$.
- Because of the symmetry, $\mathbf{A}_n$ has y-component $= 0$ and

$$\mathbf{A}_{n,x} = -m_0 + \sum_{j=1}^{n-1} \frac{\cos(\frac{2\pi j}{n}) - 1}{(2 - 2\cos(\frac{2\pi j}{n}))^{\frac{3}{2}}}.$$

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- When $n = 5$, for instance, this yields

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  and there will be an analogous m_0 value for any other $n \geq 3$.
- With this m_0, the (n-gon)$+1$ configuration becomes a neutral configuration because of the rotational symmetry.

A general result

Definition 2

A k-**dimensional regular polytope configuration** is a configuration $\mathcal{C}$ of equal masses located at the vertices of a regular polytope $\mathcal{P}$ in $\mathbb{R}^k$ such that $\mathcal{P}$ that is not contained in any hyperplane.

Theorem 3

There exists a negative mass m_0 that, when placed at the center of mass of a regular polytope configuration $\mathcal{C}$, creates a neutral configuration, $\mathcal{C}_0$.

In fact m_0 is $-\omega^2$ from the configuration $\mathcal{P}$.

A generalization

Theorem 4

Let $\mathcal{C}$ be any union of congruent regular polytope configurations in orthogonal subspaces in $\mathbb{R}^k$, all with center of mass at the origin. There exists a negative mass which, placed at the origin, makes the configuration $\mathcal{C}_0 = \mathcal{C} \cup \{\mathbf{0}\}$ neutral.

- Thanks to my colleague at Holy Cross, Andy Hwang, for suggesting this idea.

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- Thanks to my colleague at Holy Cross, Andy Hwang, for suggesting this idea.
- Also, some experiments I have done indicate that the hypothesis of congruence is not necessary, if the masses in each regular polytope configuration can be different.

Aside on regular polytopes

There is a complete classification of the regular polytopes in $\mathbb{R}^k$ up to similarity (see the classic book by Coxeter):

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3. There are 6 regular polytopes in $\mathbb{R}^4$
4. There are 3 regular polytopes in $\mathbb{R}^k$, $k \geq 5$ (simplex, hypercube, cross-polytope)

A general construction

Definition 5

Given a configuration $\mathcal{C}$ in $\mathbb{R}^k$, the **doubling of** $\mathcal{C}$ is the parametrized family of configurations for $\theta \in (0, \frac{\pi}{2})$ in $\mathbb{R}$ defined by:

$$\begin{aligned}\mathcal{D}_\theta(\mathcal{C}) = \{(\cos(\theta)\mathbf{q}, \mathbf{0}) \in \mathbb{R}^{2k} : \mathbf{q} \in \mathcal{C}\} \\ \cup \{(\mathbf{0}, \mathbf{0}) \in \mathbb{R}^{2k}\} \\ \cup \{(\mathbf{0}, \sin(\theta)\mathbf{q}) \in \mathbb{R}^{2k} : \mathbf{q} \in \mathcal{C}\}.\end{aligned}$$

How we apply this

Consider this situation:

1. $\mathcal{C}$ is a k-dimensional *regular polytope configuration*, with $n =$ number of vertices of the polytope, vertices $\mathbf{q}_i$ with $\|\mathbf{q}_i\| = 1$, all i (or one of the more general configurations from Theorem 4, with $n =$ total number of vertices), all masses $= 1$

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2. $\mathcal{C}_0 = \mathcal{C} \cup \{\mathbf{0}\}$ is an associated neutral configuration, and
3. The $\mathcal{D}_\theta(\mathcal{C})$ are $(2n + 1)$-body configurations, with all masses $= 1$ except for the central negative mass

The theorem

Theorem 6

Let $\mathcal{C}$ be a k-dimensional regular polytope configuration, or one of the more general "product regular polytope configurations" given in Theorem 4 in $\mathbb{R}^k$. Let n be the number of bodies in $\mathcal{C}$. Let $\mathcal{C}_0$ be the associated neutral configuration. Then for each $\theta \in (0, \frac{\pi}{2})$, $\mathcal{D}_\theta(\mathcal{C})$ is a central configuration with $\omega^2 = n$.

Corollary 7

The family $\mathcal{D}_\theta(\mathcal{C})$ is a continuum of inequivalent central configurations in $\mathbb{R}^{2k}$, all with the same masses.

Idea of proof

- The proof is a direct check that the c.c. conditions are satisfied for each body in the doubled configuration.
- Symmetry is used in a crucial way to simplify the calculations
- What really makes this work is that the orthogonality of the two copies of $\mathbb{R}^k$ implies

$$\|(\cos(\theta)\mathbf{q}_i, \mathbf{0}) - (\mathbf{0}, \sin(\theta)\mathbf{q}_j)\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

for all i, j.

Comments

- As before, it is easy to see that $\mathcal{D}_{\theta_1}(\mathcal{C})$ and $\mathcal{D}_{\theta_2}(\mathcal{C})$ are not equivalent if $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$.

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- In our paper, we write the continuum using a different parametrization for the doubling construction that fixes the first copy and makes $\omega^2 = \frac{n}{(1+t^2)^{3/2}}$. Equivalent to what we said here, though.

Mahalo for your attention!