Toric Surface Codes – Some New Observations

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Outline

Coding Theory Basics

- Goal: Want a provably effective way of constructing “good” linear codes over finite fields $\mathbb{F}_q$: vector subspaces C of $\mathbb{F}_q^n$ for given n

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- "Good" code means: *minimum distance* d of the code is large (for given n and $k = \dim_{\mathbb{F}_q} C$)
- Minimum distance:

$$d = \min_{x \neq y \in C} \mathrm{wt}(x - y) = \min_{x \neq 0 \in C} \mathrm{wt}(x),$$

where $\mathrm{wt}(x)$ is the Hamming weight (number of nonzero entries) – related to error-correction capacity when information is encoded to elements of C and transmitted over a noisy channel.

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- For any $e = (e_1, e_2) \in P \cap \mathbb{Z}^2$, let x^e be the corresponding monomial and write

$$(p_f)^e = (\alpha^{f_1})^{e_1} \cdot (\alpha^{f_2})^{e_2} = \alpha^{\langle f, e \rangle}.$$

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- Toric surface code $C_P(\mathbb{F}_q)$ is the linear code of block length $n = (q-1)^2$ spanned by the $(p_f)^e$ for $e \in P \cap \mathbb{Z}^2$.

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- Then $C_P(\mathbb{F}_q) = \mathrm{ev}(L)$.
- Have

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- Lots of interesting properties – higher dimensional analogs of Reed-Solomon codes

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- Can replace the set $P \cap \mathbb{Z}^m$ by an arbitrary set $S \subset \mathbb{Z}^m \cap [0, q-2]^m$.
- These "generalized toric codes" have many of the same properties

Best Known Codes From This Construction

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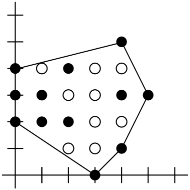
- an $m = 2$ generalized toric code over $\mathbb{F}_8$ with parameters $[49, 8, 34]$ – found by one group at MSRI-UP 2009
- different $m = 3$ generalized toric codes over $\mathbb{F}_5$ with parameters $[64, 8, 42]$ – another group at MSRI-UP 2009 and Alex Simao

Another One Found This Summer!

Over $\mathbb{F}_8$, take S given by filled in circles ($P = \mathrm{conv}(S)$ shown as well):



Get a $[49, 12, 28]$ code – best previously known for $n = 49$, $k = 12$ over $\mathbb{F}_8$ was $d = 27$.

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- Not very satisfying, though!
- There are general theoretical lower and upper bounds on d that apply to these codes (esp. work of D. Ruano, P. Beelen) *but*
- Not very easy to apply, and rarely sharp

Little-Schenk, Soprunov-Soprunova Approach

- Starting with LS, tightened and extended by SS, known that d for $C_P(\mathbb{F}_q)$ is highly correlated with $L(P) =$ *full Minkowski length* of P – the maximum number of summands in a Minkowski sum decomposition $Q = Q_1 + \cdots + Q_L$ for $Q \subseteq P$.

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- SS showed that in the plane every Minkowski-indecomposable polygon is lattice equivalent to either
  - (a) the unit lattice segment $\mathrm{conv}\{(0,0),(1,0)\}$,
  - (b) the unit lattice simplex $\mathrm{conv}\{(0,0),(1,0),(0,1)\}$, or
  - (c) the "exceptional triangle" $T_0 = \mathrm{conv}\{(0,0),(1,2),(2,1)\}$

The Soprunov-Soprunova Theorem

Theorem 1 (SS)

If q is larger than an explicit lower bound depending on $L(P)$ and the area of P, then

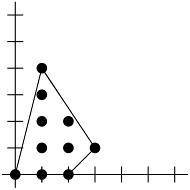
$$d(C_P(\mathbb{F}_q)) \geq (q-1)^2 - L(P)(q-1) - \lfloor 2\sqrt{q} \rfloor + 1, \tag{1}$$

and if no maximally decomposable $Q \subset P$ contains an exceptional triangle, then

$$d(C_P(\mathbb{F}_q)) \geq (q-1)^2 - L(P)(q-1). \tag{2}$$

An Example

Say $P = \mathrm{conv}\{(0,0),(2,0),(3,1),(1,4)\}$:

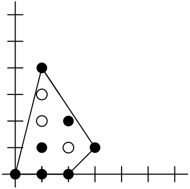


Have $L(P) = 4$, and P contains just one Minkowski sum of 4 indecomposable polygons, namely the line segment $Q = \mathrm{conv}\{(1,0),(1,4)\}$. Expect for q sufficiently large,

$$d(C_P(\mathbb{F}_q)) = (q-1)^2 - 4(q-1).$$

Example, Continued

Now, study $C_S(\mathbb{F}_q)$ for S contained in P from before:



What happens? $k = 7$ only (not $k = 10$), and ...

Example, Continued

$$
\begin{array}{rcl}
d(C_S(\mathbb{F}_7)) = 18 & \text{vs.} & 6^2 - 4 \cdot 6 = 12 \\
d(C_S(\mathbb{F}_8)) = 33 & \text{vs.} & 7^2 - 4 \cdot 7 = 21 \\
d(C_S(\mathbb{F}_9)) = 32 & \text{vs.} & 8^2 - 4 \cdot 8 = 32 \\
d(C_S(\mathbb{F}_{11})) = 70 & \text{vs.} & 10^2 - 4 \cdot 10 = 60 \\
d(C_S(\mathbb{F}_{13})) = 96 & = & 12^2 - 4 \cdot 12 = 96 \\
d(C_S(\mathbb{F}_{16})) = 165 & = & 15^2 - 4 \cdot 15 = 165 \\
d(C_S(\mathbb{F}_{17})) = 192 & = & 16^2 - 4 \cdot 16 = 192 \\
d(C_S(\mathbb{F}_{19})) = 270 & \text{vs.} & 18^2 - 4 \cdot 18 = 252 \\
d(C_S(\mathbb{F}_q)) & = & (q-1)^2 - 4(q-1) \quad \text{all } q \geq 23(?)
\end{array}
$$

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$$\mathrm{ev}(x(y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0))$$

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for all $a_i \in \mathbb{F}_q$.
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- Some of those quartic polynomials factor completely as $(y-\beta_1)\cdots(y-\beta_4)$ for $\beta_j \in \mathbb{F}_q^*$, so $4(q-1)$ zeroes in $(\mathbb{F}_q^*)^2$.
- Key point is: In $\mathbb{F}_q$ for q sufficiently large, there are *also* polynomials of the form $y^4 + a_1y + a_0$ that factor completely with distinct nonzero roots.

Families of Polynomials

Consider any linear family $\mathcal{F}$ of polynomials of the form

$$f(u) = u^\ell + t_1 u^{k_1} + \cdots + t_{m-1} u^{k_{m-1}} + t_m \tag{3}$$

in $\mathbb{F}_q[u]$, where

1. $p > \ell$,
2. the exponents $\ell > k_1 > \cdots > k_{m-1} > k_m = 0$ are fixed,
3. the coefficients t_i, $1 \leq i \leq m$ run over the finite field $\mathbb{F}_q$, and
4. the $\ell, k_1, \ldots, k_{m-1}$ are *not* all multiples of some fixed integer $j > 1$.

Factorization Patterns

- Say that a polynomial $f(u)$ of degree ℓ has factorization pattern
$$\lambda = 1^{a_1} 2^{a_2} \cdots \ell^{a_\ell},$$
where $\sum_{i=1}^{\ell} a_i \cdot i = \ell$, if in $\mathbb{F}_q[u]$, $f(u)$ factors as a product of a_i irreducible factors of degree i (not necessarily distinct) for each $i = 1, \ldots, \ell$.

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- Let
$$T(\lambda) = \frac{1}{a_1! \cdots a_\ell! 1^{a_1} \cdots \ell^{a_\ell}}$$
be the proportion of elements of the symmetric group S_ℓ with cycle decomposition of shape λ.

Cohen's Theorem

Then S. Cohen proved the following statement in 1972:

Theorem 2

Let $\mathcal{F}$ satisfy the conditions above, and let $\mathcal{F}_\lambda$ be the subset of $\mathcal{F}$ consisting of polynomials with factorization pattern λ in $\mathbb{F}_q[u]$. Then for all q sufficiently large,

$$|\mathcal{F}_\lambda| = T(\lambda)q^m + O\left(q^{m-\frac{1}{2}}\right)$$

where the implied constant depends only on ℓ.

Usually applied to produce *irreducibles* of given shapes; we want to apply it to get *"completely reducibles"*.

Distinct Roots

- We want to study factorizations of shape $\lambda = \lambda_0 := 1^\ell$ where, in addition,

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- Elements of $\mathcal{F}$ with repeated roots (possibly in some extension of $\mathbb{F}_q$) correspond to $\mathbb{F}_q$-rational points

$$(t_1, \ldots, t_m) \subset \mathcal{D}_{\mathcal{F}},$$

where $\mathcal{D}_{\mathcal{F}} = V(\Delta_{\mathcal{F}})$ and

$$\Delta_{\mathcal{F}} = \text{resultant}(f(u), f'(u), u)$$

is the *discriminant* of the family.

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- However, when the characteristic p is large enough, it is known that when the conditions above hold on $\mathcal{F}$, $\mathcal{D}_{\mathcal{F}}$ can have at most one irreducible component other than the hyperplane $V(t_m)$.
- By a general bound of Ghorpade-Lachaud, it follows that

$$|D_{\mathcal{F}}(\mathbb{F}_q)| \leq \delta \pi_{m-1},$$

where $\pi_{m-1} = |\mathbb{P}^{m-1}(\mathbb{F}_q)| = q^{m-1} + q^{m-2} + \cdots + q + 1$, and $\delta = \deg \Delta_{\mathcal{F}} \leq 2\ell - 2$.

Existence of Completely Reducibles

Corollary 3

If $p > \ell$ and $q = p^h$ is sufficiently large, there exist elements of the family $\mathcal{F} \subset \mathbb{F}_q[u]$ with factorization pattern $\lambda_0 = 1^\ell$ in which the irreducible factors are distinct, and for which all the roots are nonzero.

Proof.

The first part of this comes from comparing the orders of growth of the various terms in Cohen and Ghorpade-Lachaud. The last part of this is clear since if any of the roots is zero, then the coefficient $t_m = 0$, and the locus where that is true has dimension $m - 1$. □

First Main Theorem

Theorem 4

Let P have full Minkowski length $L(P) = \ell$ from a unique $Q \subset P$ lattice equivalent to ℓI for a primitive lattice segment. Let $S \subset Q \cap \mathbb{Z}^2$ correspond to a family $\mathcal{F}$ such that

1. *S contains the endpoints of Q, and*
2. *The k_i and ℓ are not all multiples of any fixed integer $j > 1$.*

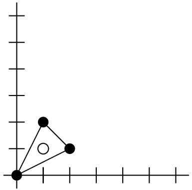
Then for all primes p sufficiently large and all $h \geq 1$, letting $q = p^h$, we have

$$d(C_S(\mathbb{F}_q)) = d(C_P(\mathbb{F}_q)) = (q-1)^2 - \ell(q-1).$$

Moreover, for all q, there exists $h \geq 1$ such that the same statement is true if we replace q by q^h.

The Exceptional Triangle

The first main theorem only applies in case there is a unique maximally decomposable Q *not* containing T_0:



Let S consist of the three boundary lattice points. Question: How do $d(C_{T_0}(\mathbb{F}_q))$ and $d(C_S(\mathbb{F}_q))$ compare?

Some Experimental Results

$$
\begin{aligned}
d(C_S(\mathbb{F}_7)) &= 27 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_7)) = 27\\
d(C_S(\mathbb{F}_8)) &= 42 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_8)) = 40\\
d(C_S(\mathbb{F}_9)) &= 56 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_9)) = 52\\
d(C_S(\mathbb{F}_{11})) &= 90 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{11})) = 85\\
d(C_S(\mathbb{F}_{13})) &= 126 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{13})) = 126\\
d(C_S(\mathbb{F}_{16})) &= 207 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{16})) = 204\\
d(C_S(\mathbb{F}_{17})) &= 240 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{17})) = 235\\
d(C_S(\mathbb{F}_{19})) &= 300 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{19})) = 300\\
d(C_S(\mathbb{F}_{23})) &= 462 \quad \text{vs.} \quad d(C_{T_0}(\mathbb{F}_{23})) = 454.
\end{aligned}
$$

Are there arbitrarily large q with $d(C_S) > d(C_{T_0})$ and also with $d(C_S) = d(C_{T_0})$?

The Corresponding Curves

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- Note total degree is ≤ 3 – if $ABD \neq 0$, the variety is irreducible, hence a curve of (arithmetic) genus 1. The family contains nodal cubics; smooth ones are *elliptic curves*.
- To understand d for corresponding codes, need to know how many $\mathbb{F}_q$-rational points they can have

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- For instance, at $[X : Y : Z] = [1 : 0 : 0]$, the tangent line is $Y = 0$, and this meets curve with multiplicity 3 – a "flex tangent."
- Flexes $\Leftrightarrow$ *points of order 3* in the group law, and *the three points at infinity form a subgroup of order 3*

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- For a Weierstrass cubic $u^2 = v^3 + \alpha v + \beta$,

$$j = 1728 \frac{4\alpha^3}{4\alpha^3 + 27\beta^2}.$$

- Curves from S with $ABD \neq 0$ always correspond to smooth elliptic curves with $j = 0$

Supersingular Curves

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- There are many equivalent characterizations of this property
- For us, the one that is most relevant (because it directly says someting about numbers of $\mathbb{F}_{p^h}$-rational points) is that the *trace of Frobenius* is zero.

Supersingular Curves

This implies that for E a supersingular curve,

$$|E(\mathbb{F}_{p^h})| = \begin{cases} p^h + 1 & h \text{ odd} \\ p^h + 1 + 2p^{h/2} & \text{if } h \equiv 2 \bmod 4 \\ p^h + 1 - 2p^{h/2} & \text{if } h \equiv 0 \bmod 4. \end{cases}$$

In other words, supersingular elliptic curves defined over $\mathbb{F}_p$ achieve the Hasse-Weil *upper* bound over $\mathbb{F}_{p^h}$ when $h \equiv 2 \bmod 4$. On the other hand, they achieve the Hasse-Weil *lower* bound over $\mathbb{F}_{p^h}$ when $h \equiv 0 \bmod 4$.

Second Main Theorem

Theorem 5

Let p be odd and $p \equiv 2 \bmod 3$. Then

$$d(C_S(\mathbb{F}_p)) = (p-1)^2 - (p-1) > d(C_{T_0}(\mathbb{F}_p)).$$

Proof. The elliptic curves from S are supersingular, so all of the codewords of $C_S(\mathbb{F}_p)$ obtained from evaluation of $Axy^2 + Bxy^2 + D$ with $ABD \neq 0$ will have weight

$$(p-1)^2 - (p+1-3) > (p-1)^2 - (p-1).$$

On the other hand, there are also codewords of weight $(p-1)^2 - (p-1)$ from polynomials with one coefficient equal to zero. Those give the minimum weight words in this case.

Proof, Concluded

By a theorem of Waterhouse, there are elliptic curves over $\mathbb{F}_p$ with

$$|E(\mathbb{F}_p)| = p + 1 + t$$

for all integers t with $t \leq \lfloor 2\sqrt{p} \rfloor$ and $\gcd(t, p) = 1$ (as well as some other possibilities). By the universality of our family for curves with nontrivial 3-torsion, there will be curves here with $p + 1 + t$ points rational over $\mathbb{F}_p$ if t is the *largest* integer satisfying $t \leq \lfloor 2\sqrt{p} \rfloor$, t prime to p, and such that $3|(p + 1 + t)$. These give codewords of considerably smaller weight, close to

$$(p-1)^2 - (p + 1 + 2\sqrt{p} - 3).$$

So d for the code from S will be strictly larger than d for the code from T_0 for all such p. $\square$

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- Go back and look at the experimental data from before! For instance $p = 23$ vs. $p = 19$.
- There are similar patterns for the C_P and C_S codes from all polygons where the Minkowski-decomposable $Q \subset P$ of maximal length contains a term lattice equivalent to T_0.

Conclusion

There are contributions both from

1. geometry of P, S, Minkowski decompositions, etc., *and*
2. arithmetic of rational points of curves over $\mathbb{F}_q$

to the minimum distance of generalized toric surface codes.
Very subtle and interesting phenomena!