Rational Approximation on the Unit Sphere in $\mathbb{C}^2$

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Abstract

For $X$ a compact subset of the unit sphere $\partial B$ in $\mathbb{C}^2$, we seek conditions implying that $R(X) = C(X)$. We conjecture an analogue of the Hartogs-Rosenthal theorem on rational approximation in the plane: if $X \subset \partial B$ is rationally convex and the three-dimensional measure of $X$ is zero, then $R(X) = C(X)$. We make several contributions to the study of this conjecture, and establish rational approximation on certain Lipschitz graphs lying in $\partial B$. In section 3, we study algebras on certain plane sets with application to approximation on $\partial B$. In section 4, we weaken the Lipschitz condition, used in section 2, to a Hölder condition.

1. Introduction

For a compact set $X \subset \mathbb{C}^n$, we denote by $R(X)$ the closure in $C(X)$ of the set of rational functions holomorphic in a neighborhood of $X$. We are interested in finding conditions on $X$ that imply that $R(X) = C(X)$, i.e. that each continuous function on $X$ is the uniform limit of a sequence of rational functions holomorphic in a neighborhood of $X$.

When $n = 1$, the theory of rational approximation is well developed. Examples of sets without interior for which $R(X) \neq C(X)$ are well-known, the “Swiss cheese” being a prime example. On the other hand, the Hartogs-Rosenthal theorem states that if the two-dimensional Lebesgue measure of $X$ is zero, then $R(X) = C(X)$.

In higher dimensions, there is an obstruction to rational approximation that does not appear in the plane. For $X \subset \mathbb{C}^n$, we denote by $\hat{X}_r$ the rationally convex hull of $X$,

2000 Mathematics Subject Classification. Primary 32E30, Secondary 46J10.
which can be defined as the set of points \( z \in \mathbb{C}^n \) such that every polynomial \( Q \) with \( Q(z) = 0 \) vanishes at some point of \( X \). The condition \( X = \hat{X}_r \) (\( X \) is rationally convex) is both necessary for rational approximation and difficult to establish, in practice, when \( n > 1 \); in the plane, every compact set is rationally convex.

We will consider primarily subsets of the unit sphere \( \partial B \) in \( \mathbb{C}^2 \). We have been motivated by a desire to obtain an analogue of the Hartogs-Rosenthal theorem in this setting. R. Basener \([5]\) has given examples of rationally convex sets \( X \subset \partial B \) for which \( R(X) \neq C(X) \); his examples have the form \( \{(z, w) \in \partial B : z \in E\} \), where \( E \subset \mathbb{C} \) is a suitable Swiss cheese. These sets have the property that \( \sigma(X) > 0 \), where \( \sigma \) is three-dimensional Hausdorff measure on \( \partial B \). It is reasonable to conjecture that if \( X \) is rationally convex, and \( \sigma(X) = 0 \), then \( R(X) = C(X) \). This paper contains several contributions to the study of this question.

In the second section we employ a construction of Henkin \([10]\). For a measure \( \mu \) supported on \( \partial B \) orthogonal to polynomials, Henkin produced a function \( K_{\mu} \in L^1(d\sigma) \), satisfying 
\[
\overline{\partial} K_{\mu} = -4\pi^2 \mu.
\]
Lee and Wermer established that if \( X \subset \partial B \) is rationally convex, and \( \mu \in R(X)\perp \) (i.e., \( \int g \, d\mu = 0 \) for all \( g \in R(X) \)), then \( K_{\mu} \) extends holomorphically to the unit ball. We show that if the extension belongs to the Hardy space \( H^1(B) \), then \( \mu \) must be the zero measure. Under an assumption on the size of the rational hull of small tubular neighborhoods of \( X \), which we call the hull-neighborhood property, we are able to show that \( K_{\mu} \) satisfies a certain boundedness condition (see Lemma 2.4 below). From this we deduce (in the proof of Theorem 2.5 below) that \( K_{\mu} \in H^1(B) \) if \( X \) is a subset of a Lipschitz graph lying in \( \partial B \). Thus in this case the only measure \( \mu \in R(X)\perp \) is the zero measure, and so \( R(X) = C(X) \). In section 4 we show how the same result can be established for graphs of Hölder functions. Also in section 2, we give an example of a class of sets satisfying the hull-neighborhood property.

In the third section we study the algebra generated by \( R(E) \) and a smooth function \( f \) on a plane set \( E \), and show that if this algebra has maximal ideal space \( E \) but does not contain all continuous functions on \( E \), then there is a subset \( E_0 \) of \( E \) on which \( f \in R(E_0) \) and \( R(E_0) \neq C(E_0) \). We then use this result to establish rational approximation on certain graphs lying in \( \partial B \).
We use the following notation in addition to that already introduced: $B$ will denote the unit ball in $\mathbb{C}^2$, coordinates of points in $\mathbb{C}^2$ will either be denoted using subscripts, such as $z = (z_1, z_2)$ or by $p = (z, w)$, according to the context. $\pi$ will denote projection to the first coordinate, i.e., $\pi(z, w) = z$. If $z, \zeta$ are points in $\mathbb{C}^2$, $\langle z, \zeta \rangle$ will denote the usual Hermitian inner product of $z$ and $\zeta$.

2. Rational Approximation and the Henkin transform

A basic tool of approximation theory in the plane is the Cauchy transform $\hat{\mu}$ of a measure $\mu$. If $\mu$ is a finite complex measure with compact support,

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z},$$

The Cauchy transform $\hat{\mu}(z)$ is integrable with respect to Lebesgue measure $m$ on the plane, is analytic in $z$ off the support of $\mu$, and satisfies the fundamental relation

$$\frac{\partial \hat{\mu}}{\partial \bar{z}} = -\pi \mu$$

in the sense of distributions, i.e.,

$$\int \phi \ d\mu = \frac{1}{\pi} \int_{\partial} \frac{\partial \phi}{\partial z} \hat{\mu} \ dm. \quad (1)$$

In [10], Henkin studied global solutions to the inhomogeneous tangential Cauchy-Riemann equations on the boundary of strictly convex domains in $\mathbb{C}^n$. His work produced transforms analogous in certain respects to the Cauchy transform. In the particular case which concerns us, the boundary of the unit ball in $\mathbb{C}^2$, Henkin introduced the kernel

$$H(z, \zeta) = \frac{\langle Tz, \zeta \rangle}{|1 - \langle z, \zeta \rangle|^2}$$

where $Tz = (\bar{z}_2, -\bar{z}_1)$. Given a measure $\mu$ supported on a set $X \subset \partial B$, the Henkin transform of $\mu$ is defined by

$$K_\mu(z) = \int_X H(z, \zeta) d\mu(\zeta).$$

Henkin showed that the integral defining $K_\mu$ converges $\sigma$-a.e on $\partial B$, $K_\mu$ is integrable with respect to $d\sigma$ on $\partial B$, and is smooth on $\partial B \setminus X$. Further, if $\mu$ satisfies the condition

$$\int_X P \ d\mu = 0, \quad \forall \text{ polynomials } P \quad (2)$$

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then $K_\mu$ satisfies
\begin{equation}
\overline{\partial}_b K_\mu = -4\pi^2 \mu.
\end{equation}
Here $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator on $\partial B$; (3) means that
\begin{equation}
\int \phi \, d\mu = \frac{1}{4\pi^2} \int_{\partial B} K_\mu \overline{\partial} \phi \wedge \omega
\end{equation}
for all functions $\phi$ smooth in a neighborhood of $\partial B$, where $\omega(z) = dz_1 \wedge dz_2$. An elementary proof of (4) is presented in H.P. Lee’s thesis [14]; Varopoulos ([19], §3.2) has also given an exposition of Henkin’s results for the case of the ball.

Note that the condition (2) that $\mu$ be orthogonal to polynomials (satisfied by all $\mu \in R(X)^\perp$) is necessary for the solution of (3), and that (3) implies that $K_\mu$ is a CR function on $\partial B \setminus X$. Lee and Wermer [15] proved that if $X$ is rationally convex, then $K_\mu$ extends holomorphically from $\partial B \setminus X$ to $B$ for any $\mu \in R(X)^\perp$:

**Theorem 2.1** Suppose $X$ is a rationally convex subset of $\partial B$. Let $\mu$ be a measure on $X$ such that $\mu \in R(X)^\perp$, and let $K_\mu$ be its Henkin transform. Then there exists a function $k_\mu$, holomorphic in a neighborhood of $\overline{B} \setminus X$, with $k_\mu = K_\mu$ on $\partial B \setminus X$.

We let $H^1(B)$ denote the Hardy space of functions $g$ holomorphic on $B$ satisfying
\[ \sup \left\{ \int_B g^{(r)} \, d\sigma : r < 1 \right\} < \infty \]
where $g^{(r)}(z) \equiv g(rz)$ for $z \in \partial B$. For $g \in H^1(B)$, $\lim_{r \to 1} g^{(r)} \equiv g^*$ exists $\sigma$-a.e on $\partial B$, and $g^{(r)} \to g^*$ as $r \to 1$ in $L^1(d\sigma)$.

**Lemma 2.2** Let $X$ be a rationally convex subset of $\partial B$ with $\sigma(X) = 0$. Let $\mu$ be a measure on $X$ with $\mu \perp R(X)$, and let $k_\mu$ be the holomorphic extension of $K_\mu$ to $B$ (as in Theorem 2.1). If $k_\mu \in H^1(B)$, then $\mu$ is the zero measure.

**Proof.** It suffices to show that $\int \phi \, d\mu = 0$ for every function $\phi \in C^1(C^2)$. Note that $\sigma(X) = 0$ implies that $k_\mu^* = K_\mu$ at $\sigma$-almost all points of $\partial B$, and so by (4)
\[ \int_X \phi \, d\mu = \frac{1}{4\pi^2} \int_{\partial B} k_\mu^* \overline{\partial} \phi \wedge \omega = \lim_{r \to 1} \frac{1}{4\pi^2} \int_{\partial B} k_\mu^{(r)} \overline{\partial} \phi \wedge \omega \]
By Stokes’ theorem, for fixed $r$
\[ \int_{\partial B} k_\mu^{(r)} \overline{\partial} \phi \wedge \omega = \int_{B} \overline{\partial}(k_\mu^{(r)} \overline{\partial} \phi \wedge \omega) = \int_{B} \overline{\partial}(k_\mu^{(r)} \overline{\partial} \phi \wedge \omega) = 0 \]
since $k_{\mu}^{(r)}$ is holomorphic in $B$. This shows that $\int \phi \, d\mu = 0$ for all $\phi \in C^1(C^2)$ and completes the proof. □

Thus to prove that $R(X) = C(X)$ for a rationally convex subset of $\partial B$ with $\sigma(X) = 0$, it suffices to show that $k_{\mu} \in H^1(B)$ for every $\mu \perp R(X)$. We will use this approach to establish rational approximation on certain subsets of $\partial B$. It should be noted that the condition that $\sigma(X) = 0$ is necessary in the preceding lemma. If $X$ is the rationally convex set constructed by Basener, $R(X) \neq C(X)$, and there exist nonzero measures $\mu \in R(X)^\perp$ for which $k_{\mu} \in H^1(B)$ ([4]).

We begin with a general estimate on the Henkin transform.

**Lemma 2.3** If $X \subset \partial B$, $\mu$ is a measure supported on $X$, and $z \in \partial B$, then

$$|K_{\mu}(z)| \leq \frac{4\|\mu\|}{\text{dist}^4(z, X)}$$

(5)

*Proof:* For any $\zeta, z \in \partial B$,

$$|z - \zeta|^2 = |z|^2 + |\zeta|^2 - 2 \text{Re}(\langle z, \zeta \rangle) = 2 \text{Re}(1 - \langle z, \zeta \rangle) \leq 2|1 - \langle z, \zeta \rangle|$$

and thus for $\zeta \in X, z \in \partial B$,

$$\text{dist}^2(z, X) \leq 2|1 - \langle z, \zeta \rangle|.$$

(6)

We obtain from this an estimate on Henkin’s kernel $H$: for $z \in \partial B, \zeta \in X$

$$|H(z, \zeta)| = \frac{|\langle Tz, \zeta \rangle|}{|1 - \langle z, \zeta \rangle|^2} \leq \frac{4|Tz||\zeta|}{\text{dist}^4(z, X)} = \frac{4}{\text{dist}^4(z, X)}$$

from which (5) follows immediately, by the definition of $K_{\mu}$. □

We would like to establish an estimate similar to (5) for the holomorphic extension $k_{\mu}$ of $K_{\mu}$ to $B$ given by Theorem 2.1 for rationally convex $X$. We shall do this for the class of sets satisfying the following strong notion of convexity with respect to rational functions:

**Definition:** Given $X \subset C^2$, let $X_\epsilon = \{z \in C^n : \text{dist}(z, X) < \epsilon\}$. We say that $X$ has the *hull-neighborhood* property (abbreviated (H-N)) if there exists $k > 0$ such that, if we put $E = \pi(X)$, we have for all $\epsilon > 0$,

$$[X_\epsilon]_r \cap \pi^{-1}(E) \subset X_{k\epsilon}.$$  

(7)
In other words, given $z \in \mathbb{C}^2$ with $\pi(z) \in \pi(X)$ and $\epsilon > 0$ so that $\text{dist}(z, X) > k\epsilon$, there exists a polynomial $Q$ with $Q(z) = 0$ whose zero set does not meet $X_\epsilon$. Since $\pi(\overline{X_r}) = \pi(X)$, it is clear that if $X$ has property (H-N), then $X$ is rationally convex. Also, for $X \subset \partial B$, $\overline{[X_\epsilon]}$ is contained in the ball of radius $1 + \epsilon$ centered at the origin, so $[X_\epsilon] \subset \mathbb{R}^2$. Therefore for $X \subset \partial B$, there exists $k > 0$ such that (7) holds for all $\epsilon > 0$ if and only if there exists $k > 0$ such that (7) holds for all sufficiently small $\epsilon$.

**Lemma 2.4** Assume $X \subset \partial B$ has property (H-N). Then there exists a constant $c$ so that for all $p \in B$ with $\pi(p) \in \pi(X)$ and all $\mu \in R(X)^\perp$, we have

\[ |k_\mu(p)| \leq \frac{c \| \mu \|}{\text{dist}^4(p, X)}. \]

**Proof:** Fix $p \in B$, set $\delta = \text{dist}(p, X)$. If $\epsilon > 0$ satisfies $k\epsilon < \delta$, then by hypothesis $p \notin \overline{[X_\epsilon]}$, so there exists a polynomial $Q$ with $Q(p) = 0$ such that the zero set $V$ of $Q$ does not meet $X_\epsilon$. Note that $k_\mu$ is continuous on $V \cap \overline{B}$ with boundary values $K_\mu$ on $V \cap \partial B$. By the maximum principle, $|k_\mu|$ attains its maximum on $V \cap \overline{B}$ at a point $p_0 \in \partial B \cap V$, and so by Lemma 2.3,

\[ |k_\mu(p)| \leq |K_\mu(p_0)| \leq \frac{4 \| \mu \|}{\text{dist}^4(p_0, X)} \leq \frac{4 \| \mu \|}{\epsilon^4}. \]

Since the preceding inequality holds whenever $k\epsilon < \delta$, we obtain (8). $\square$

Let $\Delta$ denote the closed unit disk in the complex plane. For a function defined on $\Delta$, we let $\Gamma(f) \subset \mathbb{C}^2$ denote the graph of $f$ over $\Delta$. $\text{Lip}(\Delta)$ will denote the set of Lipschitz functions on $\Delta$, i.e., those functions $f$ for which there exists a constant $M > 0$ such that $|f(z) - f(z')| \leq M|z - z'|$ for all $z, z' \in \Delta$; the least such $M$ we call the Lipschitz constant for $f$. The main result of this section is the following approximation theorem for subsets of Lipschitz graphs with the hull-neighborhood property.

**Theorem 2.5** Let $f \in \text{Lip}(\Delta)$. Assume $\Gamma(f) \subset \partial B$. If $X \subset \Gamma(f)$ has property (H-N), then $R(X) = C(X)$.

**Proof:** We will show that under the hypotheses of Theorem 2.5, $k_\mu \in H^1(B)$ for each $\mu \in R(X)^\perp$. By Lemma 2.2, since $\sigma(\Gamma(f)) = 0$ this will imply that every measure in $R(X)^\perp$ is identically zero, and hence $R(X) = C(X)$. Fix $\mu \in R(X)^\perp$, and write $k = k_\mu$. Let $(z, w)$
denote the coordinates in $\mathbb{C}^2$. We show that $k \in H^1(B)$ by estimating $k$ on the slices $z =$ constant. To do this, we first introduce some notation and prove a lemma.

For $z \in \triangle$, let $D_z = \{w : |w| < \sqrt{1 - |z|^2}\}$, and let $\gamma_z$ be the boundary of $D_z$. If $g$ is a function holomorphic in $B$ and $z \in \triangle$, we let $g_z$ denote the slice function $g_z(w) = g(z, w), w \in D_z$. If for some $s > 0$ we have $g_z \in H^s(D_z)$, i.e.,

$$ (9) \quad \sup \left\{ \int_0^{2\pi} |g_z(r \sqrt{1 - |z|^2 e^{i\theta}})|^s \, d\theta : 0 < r < 1 \right\} < \infty $$

then $g_z^s(w) = \lim_{r \to 1} g_z(rw)$ exists for almost all $w \in \gamma_z$. If in addition $g_z^s(w) \in L^1$ with respect to linear measure on $\gamma_z$, then in fact $g_z \in H^1(D_z)$ (see [8], Theorem 2.11 ) and $\int_0^{2\pi} |g(z, r \sqrt{1 - |z|^2 e^{i\theta}})| \, d\theta$ is increasing in $r$.

**Lemma 2.6** Let $X$ be a subset of $\partial B$ with $\sigma(X) = 0$. Suppose $g$ is holomorphic in a neighborhood of $\overline{B} \setminus X$, $g|_{\partial B} \in L^1(d\sigma)$, and for some $s > 0$, $g_z \in H^s(D_z)$ for almost all $z \in \triangle$. Then $g \in H^1(B)$.

**Proof:** First note that if $f$ is any positive function defined ($\sigma$ - a.e.) on $\partial B$, then (see Proposition 1.47 of [17]),

$$ (10) \quad \int_{\partial B} f \, d\sigma = \int_\triangle dm(z) \int_0^{2\pi} f_z(r \sqrt{1 - |z|^2 e^{i\phi}}) \, d\phi $$

Set $G = g|_{\partial B}$. The hypotheses imply that for $m$-almost all $z \in \triangle$, we have $G|_{\gamma_z} = g_z^s$ is defined almost everywhere and integrable with respect to linear measure on $\gamma_z$, and $g_z \in H^1(D_z)$. Thus if $r < 1$, by (10)

$$ \int_{\partial B} |g(r)| \, d\sigma = \int_\triangle dm(z) \int_0^{2\pi} |gr_z(r \sqrt{1 - |z|^2 e^{i\phi}})| \, d\phi $$

$$ \leq \int_\triangle dm(z) \int_0^{2\pi} |gr_z^s(\sqrt{1 - r^2 |z|^2 e^{i\phi}})| \, d\phi $$

The change of variables $z' = rz$ converts the last integral above to

$$ \frac{1}{r^2} \int_{\gamma_z \cdot |z' \cdot \sigma|} dm(z') \int_0^{2\pi} |G(z', \sqrt{1 - |z'|^2 e^{i\phi}})| \, d\phi \leq \frac{1}{r^2} \int_{\partial B} |G| \, d\sigma $$

again by (10). Since $G \in L^1(d\sigma)$, we find that $\int_{\partial B} |g^{(r)}| \, d\sigma$ is bounded independently of $r$, so $g \in H^1(B)$. □
By Lemma 2.6, the proof of Theorem 2.5 will be complete if we can show that for some $s > 0$, $k_z \in H^s(D_z)$ for almost all $z \in \Delta$. Fix $z \in \Delta$. We may assume $z \in \pi(X)$, for if $z \notin \pi(X)$, then $k_z$ is holomorphic in a neighborhood of the closure of $D_z$, and there is nothing to prove. If $p = (z, w)$, with $w \in D_z$, then for any $p' = (z', f(z'))$,

$$|w - f(z)| \leq |w - f(z')| + |f(z') - f(z)|$$
$$\leq |w - f(z')| + M|z - z'|$$
$$\leq \sqrt{M^2 + 1} |p - p'|$$

by the Cauchy-Schwarz inequality, and so

(11) 
$$|w - f(z)| \leq \sqrt{M^2 + 1} \text{dist}(p, X)$$

By Lemma 2.4, then

(12) 
$$|k(p)| \leq \frac{C}{\text{dist}^4(p, X)} \leq \frac{C'}{|w - f(z)|^4}$$

for some constant $C'$. Write $f(z) = \sqrt{1 - |z|^2}e^{\iota z}$. Then using (12), for $r < 1$ we obtain

$$\int_0^{2\pi} |k_z(r\sqrt{1 - |z|^2}e^{\iota z})|^{1/8} d\theta \leq \frac{C'}{(1 - |z|^2)^{1/4}} \int_0^{2\pi} \frac{1}{|r e^{\iota z} - e^{\iota \phi}|^{1/2}} d\theta$$
$$= C'' \int_0^{2\pi} \frac{1}{|r e^{\iota z} - 1|^{1/2}} d\theta$$

For $|\theta| \leq \pi/3$, $\cos(\theta) \leq 1 - \theta^2/4$, which implies

$$|1 - r e^{\iota z}|^{1/2} = [1 + r^2 - 2r \cos(\theta)]^{1/2} \geq [(1 - r)^2 + \theta^2/4]^{1/4} \geq \sqrt{\theta}/\sqrt{2}$$

It follows from this that the last integral is bounded independently of $r$, and so $k \in H^{1/8}(D_z)$ for all $z \in \Delta$. This completes the proof. □

**Remark:** The special case of Theorem 2.5 when $f$ is continuously differentiable on $\Delta$ can also be obtained as a direct consequence of Theorem 4.3 of ([2]).

We close this section by exhibiting a class of sets with the hull-neighborhood property. Recall that a real submanifold of $\mathbb{C}^n$ is said to be totally real if at each point, its tangent space contains no complex line.

**Theorem 2.7** Let $f \in C^\infty(\Delta)$, and assume $\Gamma(f)$ is a totally real submanifold of $\mathbb{C}^2$. If $X$ is a compact polynomially convex subset of $\Gamma(f)$, then $X$ has property (H-N).
Proof: For $p \in \mathbb{C}^2$, let $\delta(p) = \text{dist}(p, \Gamma(f))$. Since $\Gamma(f)$ is totally real, a result of Hörmander and Wermer ([12], or see [1], Lemma 17.2) implies that there is a neighborhood $U$ of $X$ in $\mathbb{C}^2$ such that $\delta^2$ is strictly plurisubharmonic on $U$.

Since $X$ is polynomially convex, there exists a compact polynomial polyhedron $\Pi$, $X \subset \Pi \subset U$, where $\Pi = \{|P_j| \leq 1, j = 1, \ldots, k\}$ with each $P_j$ a polynomial. We may assume that $|P_j| \leq 1/2$ on $X$, for each $j$. Define a function $\Psi$ on $\mathbb{C}^2$ by

$$\Psi = \max\{|P_1|, \ldots, |P_k|\} - \frac{3}{4}$$

Then $\Psi = 1/4$ on $\partial \Pi$ and $\Psi < 0$ on $X$.

Choose $\epsilon_0 > 0$ so small that $\Psi < 0$ on $X_{\epsilon_0}$. We will show that whenever $p \in \mathbb{C}^2$ satisfies $\pi(p) \in \pi(X)$ and $\text{dist}(p, X) > \sqrt{M^2 + 1} \epsilon$ for some $\epsilon < \epsilon_0$, where $M$ is the Lipschitz constant for $f$, then there is a polynomial $Q$ with $Q(p) = 0$ whose zero set does not meet $X_{\epsilon}$. By the remarks following the definition of (H-N), this will complete the proof.

Choose a constant $\kappa > 0$ so that $\kappa \delta^2(p) < 1/4$ for all $p \in \partial \Pi$. Then on a neighborhood $N$ of $\partial \Pi$ we have $\kappa \delta^2 < \Psi$. Define $F$ as follows:

$$F = \begin{cases} 
\max(\Psi, \kappa \delta^2) & \text{on } \Pi \cup N \\
\Psi & \text{on } \mathbb{C}^2 \setminus \Pi 
\end{cases}$$

Then $F$ is well-defined and plurisubharmonic on $\mathbb{C}^2$. For $\epsilon < \epsilon_0$ set

$$\Lambda_{\epsilon} = \{q \in \mathbb{C}^2 : F(q) \leq \kappa \epsilon^2\}$$

Then $\Lambda_{\epsilon}$ is compact, and $X_{\epsilon} \subset \Lambda_{\epsilon}$, for if $\text{dist}(q, X) < \epsilon$, then $\Psi(q) < 0$, so

$$F(q) = \kappa \delta^2(q) \leq \kappa \text{dist}^2(q, X) < \kappa \epsilon^2$$

implying $q \in \Lambda_{\epsilon}$. Also, since $F$ is plurisubharmonic, $\Lambda_{\epsilon}$ is polynomially convex (this follows from [11], Theorem 4.3.4). Suppose $p$ satisfies $\text{dist}(p, X) > \sqrt{M^2 + 1} \epsilon$. We distinguish two cases: either (1) $F(p) = \kappa \delta^2(p)$, or (2) $F(p) = \Psi(p)$. In the first case, we find as in the proof of Theorem 2.5 that if we write $p$ in coordinates as $p = (z, w)$ then $|w - f(z)| \leq \sqrt{M^2 + 1} |p - p'|$ whenever $p' \in \Gamma(f)$, implying $\text{dist}(p, X) \leq \sqrt{M^2 + 1} \delta(p)$, and so

$$F(p) \geq \frac{\kappa \text{dist}^2(p, X)}{M^2 + 1} > \kappa \epsilon^2$$
and thus $p \not\in \Lambda$. By the polynomial convexity of $\Lambda$, there exists a polynomial $Q$, nonvanishing on $\Lambda$, with $Q(p) = 0$; since $X_e \subset \Lambda$, $Q$ does not vanish on $X_e$. In the second case, we must have $\Psi(p) > 0$, and so $|P_j(p)| > 3/4$ for some $j$. Set $Q = P_j - P_j(p)$. Then $Q(p) = 0$, but since $\Psi < 0$ on $X_e$, $|P_j| < 3/4$ on $X_e$, so $Q$ cannot vanish on $X_e$. In both cases, we have found the required polynomial $Q$, and the proof is complete. □

Finally we note that the approach in this section is related to the problem of determining when $X$ is a removable singularity for integrable CR functions. In this context, we may say that $X$ is removable for $L^1$ CR functions if $X$ has the property that whenever $g \in L^1(d\sigma)$ and $\bar{\partial} g = 0$ off $X$, then $g$ extends to a function in $H^1(B)$ (see [3]). By (3), $\bar{\partial} K_\mu = 0$ off $X$ whenever $\mu \in R(X)^+$, and hence by the remarks following Lemma 2.2, $R(X) = C(X)$ for any subset of $\partial B$ with $\sigma(X) = 0$ that is removable for $L^1$ CR functions. The paper [16] contains an extensive bibliography on this question and a survey of recent results.

3. The algebra generated by $R(E)$ and a smooth function

In this section we study the algebra generated by $R(E)$ and a smooth function on a planar set $E$. We then apply our results to the question of rational approximation on certain subsets of $\partial B$.

If $\mathcal{A}$ is a uniform algebra on a compact space $X$, we write $\mathcal{M}(\mathcal{A})$ for its maximal ideal space, and view elements of $\mathcal{M}(\mathcal{A})$ as homomorphisms $m : \mathcal{A} \to \mathbb{C}$. We will identify each point $x \in X$ with the point evaluation $m_x \in \mathcal{M}(\mathcal{A})$ defined by $m_x(h) = h(x)$. When $\mathcal{A} = R(X)$ for some compact subset $X \subset \mathbb{C}^n$, then $\mathcal{M}(\mathcal{A})$ can be identified with $\tilde{X}$ via $m \in \mathcal{M}(\mathcal{A}) \to (m(z_1), \ldots, m(z_n))$ where $(z_1, \ldots, z_n)$ are the coordinate functions. This correspondence is a homeomorphism.

If $\mathcal{F}$ is a family of continuous functions on a compact space $X$, then $[\mathcal{F}]$ will denote the algebra generated by $\mathcal{F}$, i.e., the smallest closed subalgebra of $C(X)$ containing $\mathcal{F}$. In [20], J. Wermer studied the algebra $\mathcal{A} = [z, f]$ on $\Delta$ generated by the identity function $z$ and a smooth function $f$. Under the assumption that $\mathcal{M}(\mathcal{A}) = \Delta$, he showed that $\mathcal{A}$ consists of those continuous functions on $\Delta$ whose restrictions to the zero set $E$ of $\partial f/\partial \bar{z}$ lie in $R(E)$. We will make use of the following generalization of Wermer’s result due to Anderson and
Izzo ([2], Theorem 4.2):

**Lemma 3.1** Let \( \mathcal{G} \) be a collection of continuously differentiable functions on \( \triangle \), and set \( \mathcal{A} = [\mathcal{G}] \). Assume the function \( z \) lies in \( \mathcal{A} \), and that \( \mathcal{M}(\mathcal{A}) = \triangle \). Set \( T = \{ \zeta \in \triangle : \frac{\partial g}{\partial \bar{z}}(\zeta) = 0, \forall g \in \mathcal{G} \} \). Then \( \mathcal{A} = \{ g \in C(\triangle) : g|_T \in R(T) \} \).

In order to pass from algebras on compact subsets of the disk to algebras on the disk, we will need two results on extension algebras. The first is due to Bear [6]:

**Lemma 3.2** Let \( \mathcal{A}_0 \) be a uniform algebra on a compact subset \( X_0 \) of a compact space \( X \). Put \( \mathcal{A} = \{ h \in C(X) : h|_{X_0} \in \mathcal{A}_0 \} \). If \( \mathcal{M}(\mathcal{A}_0) = X_0 \), then \( \mathcal{M}(\mathcal{A}) = X \).

**Lemma 3.3** Let \( \mathcal{A}, \mathcal{A}_0, X, \) and \( X_0 \) be as in Lemma 3.2. Assume \( \mathcal{G}_0 \) is a subset of \( C(X_0) \) with \( [\mathcal{G}_0] = \mathcal{A}_0 \). Let \( \mathcal{G} \subset C(X) \) and assume (1) \( [\mathcal{G}] \) contains all continuous functions on \( X \) vanishing in a neighborhood of \( X_0 \), and (2) \( \mathcal{G}|_{X_0} = \mathcal{G}_0 \). Then \( [\mathcal{G}] = \mathcal{A} \).

**Proof:** Clearly \( \mathcal{G} \subset \mathcal{A} \), and so it suffices to show, given \( h \in \mathcal{A} \), that \( \int h \, d\mu = 0 \) for all measures \( \mu \in [\mathcal{G}]^\perp \). For any such measure the hypothesis that \([\mathcal{G}]\) contains all continuous functions vanishing near \( X_0 \) implies \( \text{supp}(\mu) \subset X_0 \). Since \( h|_{X_0} \in \mathcal{A}_0 \), we may choose a sequence \( h_j \) of polynomials in elements of \( \mathcal{G}_0 \) converging to \( h \) on \( X_0 \). By hypothesis (2), we may assume each \( h_j \) is the restriction to \( X_0 \) of an element of \([\mathcal{G}]\). Then

\[
\int_X h \, d\mu = \int_{X_0} h \, d\mu = \lim_{j \to \infty} \int_{X_0} h_j \, d\mu = 0
\]

since \( \mu \in [\mathcal{G}]^\perp \). \( \square \)

Given a compact \( E \subset \mathbb{C} \), we write \( f \in C^1(E) \) if \( f \) is the restriction to \( E \) of a function continuously differentiable in some neighborhood of \( E \).

**Theorem 3.4** Let \( E \) be a compact subset of \( \mathbb{C} \), and take \( f \in C^1(E) \). Assume \( \mathcal{M}([R(E), f]) = E \). If \( [R(E), f] \neq C(E) \), then there exists a compact subset \( E_0 \) of \( E \) such that \( R(E_0) \neq C(E_0) \) and \( f|_{E_0} \in R(E_0) \).

**Proof:** Let \( E \) and \( f \) satisfy the hypotheses of the theorem. Without loss of generality, \( E \) is a compact subset of the open unit disk. Set \( \mathcal{A} = \{ h \in C(\triangle) : h|_E \in [R(E), f] \} \). Since \( \mathcal{M}([R(E), f]) = E \) by hypothesis, Lemma 3.2 implies that \( \mathcal{M}(\mathcal{A}) = \triangle \). Fix any smooth extension of \( f \) to \( \triangle \) (we denote the extension by \( f \), also). Since \( R(E) \) is generated by the
set of functions holomorphic in a neighborhood of $E$. Lemma 3.3 implies that $\mathcal{A}$ is generated by the set $\mathcal{G}$ consisting of $f$ together with all functions smooth on $\Delta$ and holomorphic in a neighborhood of $E$. Set $E_0 = \{ \zeta \in \Delta : \partial g/\partial \zeta(\zeta) = 0, \ \forall g \in \mathcal{G}\}$. Clearly $E_0 \subset E$. By Lemma 3.1, $\mathcal{A} = \{ h \in C(X) : h|_{E_0} \in R(E_0) \}$. Since $f \in \mathcal{A}$, $f|_{E_0} \in R(E_0)$. If $R(E_0) = C(E_0)$, then $\mathcal{A} = C(X)$ and hence $[R(E), f] = C(E)$, contrary to hypothesis. □

As mentioned in the introduction, Basener gave examples of rationally convex subsets $X$ of $\partial B$ with $R(X) \neq C(X)$. To explain Basener’s construction, we recall the notion of a Jensen measure. Given a uniform algebra $\mathcal{A}$ on $X$, a probability measure $\sigma$ on $X$ is said to be a Jensen measure for $m \in \mathcal{M}(\mathcal{A})$ if for every $h \in \mathcal{A}$,

$$\log |m(h)| \leq \int_X \log |h| \, d\sigma.$$ 

If $m$ is point evaluation at some $p_0 \in X$, the point mass $\delta_{p_0}$ at $p_0$ is trivially a Jensen measure for $m$. Every Jensen measure $\sigma$ for $m$ represents $m$: $m(h) = \int h \, d\sigma$ for all $h \in \mathcal{A}$. Basener’s assumption for $X \subset \partial B$ was the following condition on $E = \pi(X)$:

(B) For all $z_0 \in E$ the only Jensen measure for $z_0$ relative to $R(E)$ is $\delta_{z_0}$.

It can be shown (see [7], Theorem 3.4.11) that (B) is equivalent to the condition that the set of functions harmonic in a neighborhood of $E$ is dense in $C(E)$. Examples of sets $E \subset \mathbb{C}$ satisfying (B) for which $R(E) \neq C(E)$ can be found in [7], p. 193 ff. and [18], §27.

Basener showed that if $X \subset \partial B$ has the form $X = \{ (z,w) \in \partial B : z \in E \}$ where $E$ is a compact subset of the open unit disk satisfying (B), then $X$ is rationally convex; in fact, his proof shows (see also [18], §19.8) that the same is true for any $X \subset \partial B$ for which $\pi(X) = E \subset \text{int}(\Delta)$ satisfies (B). Our next lemma has a similar flavor:

**Lemma 3.5** Let $E$ be a compact subset of $\mathbb{C}$ satisfying (B), and let $f \in C(E)$. Then $\mathcal{M}([R(E), f]) = E$.

This can be proved by an argument essentially the same as that of Basener mentioned above, but a simpler approach is to note that it is an immediate consequence of the following easy lemma (which strengthens Lemma 2.2 of [13]).
Lemma 3.6  Suppose $\mathcal{A}$ and $\mathcal{B}$ are uniform algebras on a compact space $X$ and $\mathcal{A} \subset \mathcal{B}$. If $x \in X$ is such that the only Jensen measure for $x$ relative to $\mathcal{A}$ is $\delta_x$, and $m \in \mathcal{M}(\mathcal{B})$ coincides with point evaluation at $x$ when restricted to $\mathcal{A}$, then $m$ is point evaluation at $x$ on all of $\mathcal{B}$. □

Proof: Let $\mu$ be a Jensen measure for $m$ (as a functional on $\mathcal{B}$). Then obviously $\mu$ is a Jensen measure for the restriction of $m$ to $\mathcal{A}$, i.e., for point evaluation at $x$ on $\mathcal{A}$. Hence by hypothesis $\mu = \delta_x$. Since $\mu$ represents $m$, we conclude that $m$ is point evaluation at $x$ on all of $\mathcal{B}$. □

If $\mathcal{A}$ is a uniform algebra on $X$, a point $p \in X$ is a peak point for $\mathcal{A}$ if there exists a function $f \in \mathcal{A}$ with $f(p) = 1$ while $|f| < 1$ on $X \setminus \{p\}$. When $X$ is a compact planar set, Bishop proved that $R(X) = C(X)$ if almost every point of $X$ is a peak point for $R(X)$.

Theorem 3.7  Let $E$ be a compact subset of $\mathbb{C}$ satisfying (B), and let $f \in C^1(E)$. If almost every point of $E$ is a peak point for $[R(E), f]$, then $[R(E), f] = C(E)$.

Proof: Suppose that $[R(E), f] \neq C(E)$. By Lemma 3.5, $\mathcal{M}([R(E), f]) = E$. We may then apply Theorem 3.4 to produce a compact subset $E_0$ of $E$ with $f|_{E_0} \in R(E_0)$ and $R(E_0) \neq C(E_0)$. If $z \in E_0$ is a peak point for $[R(E), f]$, choose $g \in [R(E), f]$ peaking at $z$. Since $g|_{E_0} \in R(E_0)$, the point $z$ is a peak point for $R(E_0)$. By Bishop’s peak-point theorem, $R(E_0) = C(E_0)$, which is a contradiction. □

Corollary 3.8  Let $E$ be a compact subset of the open unit disk satisfying (B), let $f \in C^1(E)$, and set $X = \{(z, f(z)) : z \in E\}$. If $X \subset \partial B$, then $R(X) = C(X)$.

Proof: Let $\mathcal{A}$ be the algebra on $X$ generated by $r(z)$ and $w$, where $(z, w)$ are coordinates in $\mathbb{C}^2$ and $r$ ranges over $R(E)$. Since $\mathcal{A} \subset R(X)$, it suffices to show that $\mathcal{A} = C(X)$. Moreover, $\mathcal{A}$ is isometrically isomorphic to the algebra on $E$ generated by $R(E)$ and $f$, and therefore it is enough to show $[R(E), f] = C(E)$. Each point of $\partial B$ is a peak point for polynomials, hence is a peak point for $\mathcal{A}$, and so every point of $E$ is a peak point for $[R(E), f]$. By Theorem 3.7, $[R(E), f] = C(E)$. □

It is reasonable to conjecture that Theorems 3.4 and 3.7 remain valid if the hypothesis that $f \in C^1(E)$ is replaced by the assumption that $f$ is merely continuous on $E$. We have
no proof or counterexample.

Finally, we remark that Theorem 3.7 can also be obtained in a different fashion by combining our Lemma 3.5 with Theorem 4.3 of [2].

4. Approximation on Hölder graphs

In this section we show that the hypothesis \( f \in \text{Lip}(\Delta) \) of Theorem 2.5 may be weakened to the assumption that \( f \) satisfies a Hölder condition with exponent \( \alpha, 0 < \alpha < 1 \), on \( E = \pi(X) \). That is, we assume there exists \( M \) so that for all \( z, z' \in E \),

\[
|f(z) - f(z')| \leq M |z - z'|^\alpha
\]

To establish Theorem 2.5 under the hypothesis that \( f \) satisfies (13), it suffices to show (cf. (11) in the proof of Theorem 2.5) that there exists a constant \( C \) so that for \( z \in E, w \in D_z \),

\[
|w - f(z)| \leq C \text{ dist}((z, w), X)^\alpha
\]

From (14) it follows, as in the proof of Theorem 2.5, that if \( p = (z, w) \), we have the estimate

\[
|k(p)| \leq \frac{C'}{|w - f(z)|^{1/\alpha}}
\]

from which we infer \( k \in H^{\alpha/8}(D_z) \) for all \( z \in \Delta \), completing the proof.

To establish (14), we fix \( p = (z, w) \), and take

\[ p' = (z', f(z')) \in X \]

so that \( \text{dist}(p, X) = |p - p'| \). Then

\[
|w - f(z)| \leq |w - f(z')| + |f(z') - f(z)|
\]

\[
\leq |w - f(z')| + M |z - z'|^\alpha
\]

\[
\leq (M^2 + 1)^{1/2} (|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/2}
\]

and so

\[
\frac{|w - f(z)|^{2/\alpha}}{\text{dist}^2(p, X)} \leq \frac{(M^2 + 1)^{1/\alpha} (|w - f(z')|^2 + |z - z'|^{2\alpha})^{1/\alpha}}{|w - f(z')|^2 + |z - z'|^2}
\]

Set \( x = |w - f(z')|, y = |z - z'| \). Note \( \text{dist}^2(p, X) = x^2 + y^2 \leq 4 \), since \( p, p' \) are points in the closed unit ball. The quantity

\[
G(x, y) = \frac{(x^2 + y^{2\alpha})^{1/\alpha}}{x^2 + y^2}
\]
on the right of (15) is clearly bounded on $1 \leq x^2 + y^2 \leq 4$, so to complete the proof of (14), it suffices to show that $G(x, y)$ is bounded for $x^2 + y^2 < 1$. Applying the elementary inequality $(A + B)^p \leq 2^p (A^p + B^p)$ for positive $A, B, p$, we obtain

$$(x^2 + y^{2\alpha})^{1/\alpha} \leq 2^{1/\alpha} (x^{2/\alpha} + y^2) \leq 2^{1/\alpha} (x^2 + y^2)$$

using, in the last inequality, the fact that $x < 1$. Therefore, $G(x, y) \leq 2^{1/\alpha}$ for $x^2 + y^2 < 1$, and the proof is finished.

References


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