Approximation by CR Functions on the Unit Sphere in $\mathbb{C}^2$

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Dedicated to Joe Cima on the occasion of his seventieth birthday

Abstract. For a smoothly bounded relatively open subset Ω of the unit sphere in $\mathbb{C}^2$ we derive, using a kernel $H(\zeta, z)$ introduced by G. Henkin, an analogue of the Cauchy-Green formula in the plane:

$$\phi(z) = A(z) + \frac{1}{4\pi^2}\int_\Omega \overline{\partial}\phi(\zeta)H(\zeta,z)\omega(\zeta) - \frac{1}{4\pi^2}\int_{\partial\Omega}\phi(\zeta)H(\zeta,z)\omega(\zeta),\ z \in \Omega$$

valid for $\phi \in C^1(\overline{\Omega})$, where A is a CR function on Ω and $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$. We employ this formula to study rational approximation on compact subsets K of S, by using it to estimate the distance in $C(K)$ of ϕ to the CR functions on a neighborhood Ω of K. This requires an examination of the integral over $\partial\Omega$ appearing in the above formula, which we denote by $F_\Omega(z)$; in some circumstances we can show that F_Ω also defines a CR function on Ω, and thereby estimate the distance of ϕ to the CR functions on Ω in terms of $X(\phi)$, where X is the tangential Cauchy-Riemann operator on S. For certain K we can show that $R(K) = C(K)$ by showing that the distance of $\overline{z}_j$ to the CR functions on Ω tends to zero as Ω shrinks to K.

1. Introduction

Let D be a smoothly bounded domain in the complex plane and let f be a function in $C^1(\overline{D})$. The *Cauchy-Green formula* for D allows us to represent the value of f at a point $z \in D$ in terms of the values of f on ∂D and of the one-form $\overline{\partial} f$ on D:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_D \overline{\partial} f(\zeta) \wedge \frac{d\zeta}{\zeta - z}. \tag{1.1}$$

We note the following consequences of (1.1). Let μ be a finite complex measure on $\mathbb{C}$, of compact support. The *Cauchy transform* $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(\zeta) = \int \frac{d\mu(z)}{z - \zeta},\ \ z \in \mathbb{C}.$$

Given a function $f \in C_0^1(\mathbb{C})$, we then have

$$\int_{\mathbb{C}} f d\mu = \frac{1}{2\pi i}\int_{\mathbb{C}} \hat{\mu}(\zeta)\overline{\partial} f(\zeta) \wedge d\zeta. \tag{1.2}$$

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Equation (1.2) follows from (1.1) by letting D be a large disk, multiplying by μ and integrating over $\mathbb{C}$.

Given a compact set $K \subset \mathbb{C}$, $C(K)$ is the Banach algebra of continuous functions on K with norm $\|f\|_K = \max\{|f(z)| : z \in K\}$. For a smoothly bounded plane domain D, let $A(D)$ denote the space of functions $g \in C(\overline{D})$ with g holomorphic on D. For $f \in C^1(\partial D)$, set

$$F(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in D.$$

Then F extends to $\overline{D}$ (see [**3**]) with $F \in C^1(\overline{D})$, and in particular $F \in A(D)$. The formula (1.1) gives, for $z \in D$,

$$(1.3) \qquad |f(z) - F(z)| = \left| \frac{1}{2\pi i} \int_D \frac{\partial f}{\partial \overline{\zeta}}(\zeta) \, \frac{d\overline{\zeta} \wedge d\zeta}{\zeta - z} \right| \leq \frac{1}{\pi} \max_{\overline{D}} \left| \frac{\partial f}{\partial \overline{\zeta}} \right| \cdot \left[\int_D \frac{dm_2(\zeta)}{|\zeta - z|} \right],$$

where $m_2(\zeta)$ is two-dimensional Lebesgue measure. An inequality of Mergelyan (see [**5**]) states that

$$(1.4) \qquad \int_D \frac{dm_2(\zeta)}{|\zeta - z|} \leq 2\sqrt{\pi} \cdot \sqrt{m_2(D)}.$$

It follows by continuity of f and F on $\overline{D}$ that

$$(1.5) \qquad \operatorname{dist}(f, A(D)) \equiv \inf\{\|f - \Psi\|_{\overline{D}} : \psi \in A(D)\} \leq \frac{2}{\sqrt{\pi}} \cdot \max_{\overline{D}} \left| \frac{\partial f}{\partial \overline{\zeta}} \right| \sqrt{m_2(D)}.$$

For a compact set $K \subset \mathbb{C}$, let $R(K)$ be the closure in $C(K)$ of rational functions holomorphic in a neighborhood of K. As a corollary of (1.5) we obtain the following classical result.

THEOREM 1.1 (The Hartogs-Rosenthal Theorem). *Let K be a compact set in $\mathbb{C}$ with $m_2(K) = 0$. Then $R(K) = C(K)$.*

PROOF. Fix $f \in C^1(\mathbb{C})$. Choose a sequence of open, smoothly bounded sets $\{D_n\}$ decreasing to K. Fix $\epsilon > 0$. By (1.5), for each n there exists $F_n \in A(D_n)$ with

$$\|f - F_n\|_{\overline{D_n}} < \frac{2}{\sqrt{\pi}} \cdot \max_{\overline{D_n}} \left| \frac{\partial f}{\partial \overline{\zeta}} \right| \sqrt{m_2(D_n)} + \epsilon.$$

By Runge's Theorem, there exists $r_n \in R(K)$ such that $\|F_n - r_n\|_K < \epsilon$. It follows that

$$\|f - r_n\|_K < \frac{2}{\sqrt{\pi}} \cdot \max_{\overline{D_1}} \left| \frac{\partial f}{\partial \overline{\zeta}} \right| \sqrt{m_2(D_n)} + 2\epsilon.$$

Since $m_2(D_n) \to m_2(K) = 0$ as $n \to \infty$, we get $\|f - r_n\|_K < 3\epsilon$ for n sufficiently large. As ϵ was arbitrary, $f|_K \in R(K)$. Restrictions of functions in $C^1(\mathbb{C})$ to K are dense in $C(K)$, so $R(K) = C(K)$. $\square$

Our goal in this paper is to study the above situation when the complex plane is replaced by the unit sphere S in $\mathbb{C}^2$, and analytic functions on a domain in $\mathbb{C}$ are replaced by CR functions on a domain on S. Our work uses the kernel introduced by G. Henkin in [**6**]. Related integral formulas are given by Chen and Shaw in [**4**].

In section 2 we describe Henkin's kernel and the analogues of formulas (1.1) and (1.2) on the sphere S. In section 3 we give a Cauchy-Green formula for a smoothly bounded domain on S. The remainder of the paper is devoted to a study of the integrals appearing in this formula and applications to approximation results.

2. Henkin's kernel for S

We denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^2$, and let $S = \partial\mathbb{B}$ be the unit sphere. Points in $\mathbb{C}^2$ will normally be written as $z = (z_1, z_2)$ or $\zeta = (\zeta_1, \zeta_2)$, and the Hermitian inner product as $\langle \zeta, z\rangle \equiv \zeta_1\overline{z}_1 + \zeta_2\overline{z}_2$. The standard invariant three-dimensional measure on S is written σ (note: σ is not normalized). We will also frequently make use of the 2-form $\omega(\zeta) \equiv d\zeta_1 \wedge d\zeta_2$. We denote by $A(\mathbb{B})$ the *ball algebra* consisting of all functions continuous on the closure of $\mathbb{B}$ and holomorphic on $\mathbb{B}$. If Ω is an open subset of S, we say $g \in C^1(\Omega)$ is a *smooth CR function* on Ω, if $Xg = 0$ on Ω, where X is the tangential Cauchy-Riemann operator on S. Expressed in the coordinates of $\mathbb{C}^2$,

$$X = z_2 \frac{\partial}{\partial \overline{z}_1} - z_1 \frac{\partial}{\partial \overline{z}_2}.$$

Any function holomorphic in a neighborhood of Ω in $\mathbb{C}^2$ is a smooth CR function on Ω. Using the fact that for smooth ϕ

$$\overline{\partial}\phi \wedge \omega = 2(X\phi)\, d\sigma \tag{2.1}$$

as measures on S, we see using Stokes' theorem that g is a smooth CR function on Ω if and only if

$$\int_\Omega \overline{\partial}\phi \cdot g \wedge \omega = 0 \tag{2.2}$$

for every $\phi \in C^\infty(\mathbb{C}^2)$ whose support meets S in a compact subset of Ω. We say that $g \in C(\Omega)$ is a *continuous CR function* on Ω, and write $g \in CR(\Omega)$, if (2.2) holds for all such ϕ. It follows easily from (2.2) that $CR(\Omega)$ is closed under uniform convergence on compact subsets of Ω. Since every $g \in A(\mathbb{B})$ is a uniform limit of polynomials on S, it follows that $g \in CR(S)$ if g is in the ball algebra and that $g \in A(\mathbb{B})$ is a smooth CR function on Ω whenever $g\big|_\Omega \in C^1(\Omega)$. For a compact $K \subset S$, let $CR(K)$ be the uniform closure in $C(K)$ of functions that are CR in some relatively open neighborhood of K in S.

As a replacement for the Cauchy kernel $(\zeta - z)^{-1}$ Henkin gave the kernel

$$H(\zeta, z) = \frac{\overline{\zeta}_1\overline{z}_2 - \overline{\zeta}_2\overline{z}_1}{|1 - \langle z, \zeta\rangle|^2}, \quad \zeta, z \in S.$$

On $S \times S, H$ is real-analytic off the diagonal $\{\zeta = z\}$, where $\langle z, \zeta\rangle = 1$. We consider as an analogue of the Cauchy transform $\hat{\mu}$ of a measure μ the transform

$$K_\mu(\zeta) = \int_S H(\zeta, z)\, d\mu(z), \quad \zeta \in S,$$

for a measure μ on S. The integral defining K_μ converges absolutely a.e-$d\sigma$ on S, and $K_\mu \in L^1(S, \sigma)$. Under the assumption that μ is a measure on S orthogonal to the restriction to S of every holomorphic polynomial on $\mathbb{C}^2$, Henkin proves

$$\int_S \phi\, d\mu = \frac{1}{4\pi^2} \int_S \overline{\partial}\phi \wedge K_\mu \cdot \omega, \quad \phi \in C^1(S). \tag{2.3}$$

The orthogonality assumption on μ is necessary since the right-hand side of (2.3) vanishes if ϕ is the restriction to S of a polynomial.

We record below some useful properties of H and K_μ:

(i) $H(\zeta, z) = -H(z, \zeta)$;
(ii) If U is a unitary transformation of $\mathbb{C}^2$ with $\det(U) = 1$, $H(U\zeta, Uz) = H(\zeta, z)$;
(iii) $X[H(\zeta, z)] = -(1 - \langle z, \zeta \rangle)^{-2}$ (differentiation is in the ζ variable);
(iv) $K_\mu \in CR(S \setminus \mathrm{supp}(\mu))$.

Properties (i) - (iii) are routine computations, while (iv) follows immediately from (2.3) and the definition (2.2) by taking by taking ϕ supported on an open set disjoint from $\mathrm{supp}(\mu)$.

As an analogue of the Cauchy-Green formula (1.1), we have the following representation on S: given $\phi \in C^1(S)$, there exists a function $\Phi \in A(\mathbb{B})$ such that

$$\phi(z) = \Phi(z) + \frac{1}{4\pi^2} \int_S \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\, \omega(\zeta), \quad z \in S. \tag{2.4}$$

This formula, which we call the *Cauchy-Green formula on S* follows directly from (2.3) as follows: set

$$K(z) = \int_S \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\, \omega(\zeta).$$

Choose a measure μ on S with μ orthogonal to polynomials. Then

$$\int_S K(z)\, d\mu(z) = \int_S \left[\int_S H(\zeta, z)\, d\mu(z)\right] \overline{\partial}\phi(\zeta) \wedge \omega(\zeta) = \int_S \overline{\partial}\phi(\zeta) \wedge K_\mu(\zeta)\, \omega(\zeta).$$

By (2.3),

$$\int_S (4\pi^2\phi - K)\, d\mu = 0.$$

Since this holds for every μ orthogonal to polynomials, the Hahn-Banach Theorem implies that $4\pi^2\phi - K$ belongs to the uniform closure of polynomials on S, and so is the restriction to S of a function in $A(\mathbb{B})$, giving (2.4). Other proofs of (2.4) are given in [**4**] and [**1**].

3. The Cauchy-Green formula for a domain on S

Let Ω^+ be a smoothly bounded domain on S, and fix a function $\phi \in C^1(\overline{\Omega^+})$. We wish to generalize formula (2.4) to this situation. Let Ω^- denote the complement of $\overline{\Omega^+}$ on S. Note that $\partial\Omega^- = -\partial\Omega^+$ as oriented manifolds.

THEOREM 3.1. *For $z \in \Omega^+$ we have*

$$\phi(z) = A(z) + \frac{1}{4\pi^2} \int_{\Omega^+} \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\, \omega(\zeta) - \frac{1}{4\pi^2} \int_{\partial\Omega^+} \phi(\zeta) H(\zeta, z)\, \omega(\zeta),$$

where $A \in CR(\Omega^+)$.

PROOF. We form a smooth extension $\tilde{\phi}$ of ϕ to S. By (2.4) there exists $\tilde{\Phi} \in A(\mathbb{B})$ such that for $z \in S$,

$$\tilde{\phi}(z) = \tilde{\Phi}(z) + \frac{1}{4\pi^2} \int_S \overline{\partial}\tilde{\phi}(\zeta) \wedge H(\zeta, z) \cdot \omega(\zeta)$$

and so

$$\tilde{\phi}(z) = \tilde{\Phi}(z) + \frac{1}{4\pi^2} \int_{\Omega^+} \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\, \omega(\zeta) + \frac{1}{4\pi^2} \int_{\Omega^-} \overline{\partial}\tilde{\phi}(\zeta) \wedge H(\zeta, z)\, \omega(\zeta). \tag{3.1}$$

We rewrite the last term of this equation as follows: for $z \in \Omega^+$ fixed, the function $\zeta \mapsto H(\zeta, z)$ is smooth on $\overline{\Omega^-}$, and so we may use Stokes' theorem to write

$$\begin{aligned}
\int_{\partial\Omega^-} \tilde{\phi}(\zeta)H(\zeta,z)\,\omega(\zeta) &= \int_{\Omega^-} d[\tilde{\phi}(\zeta)H(\zeta,z)\,\omega(\zeta)] \\
&= \int_{\Omega^-} \overline{\partial}[\tilde{\phi}(\zeta)H(\zeta,z)\,\omega(\zeta)] \\
&= \int_{\Omega^-} \overline{\partial}\tilde{\phi}(\zeta)\wedge H(\zeta,z)\,\omega(\zeta) + \int_{\Omega^-} \tilde{\phi}(\zeta)\,\overline{\partial}H(\zeta,z)\wedge\omega(\zeta).
\end{aligned}$$

Using (2.1) and property (iii) of the Henkin kernel the second integral on the right-hand side of the last equation can be expressed as

$$-2\int_{\Omega^-} \tilde{\phi}(\zeta)\frac{d\sigma(\zeta)}{(1-\langle z,\zeta\rangle)^2}.$$

Therefore,

$$\int_{\Omega^-} \overline{\partial}\tilde{\phi}(\zeta)\wedge H(\zeta,z)\,\omega(\zeta) = \int_{\partial\Omega^-} \tilde{\phi}(\zeta)H(\zeta,z)\omega(\zeta) + 2\int_{\Omega^-} \tilde{\phi}(\zeta)\frac{d\sigma(\zeta)}{(1-\langle z,\zeta\rangle)^2}, \tag{3.2}$$

and so we may rewrite (3.1) as

$$\tilde{\phi}(z) = A(z) + \frac{1}{4\pi^2}\int_{\Omega^+} \overline{\partial}\phi(\zeta)\wedge H(\zeta,z)\omega(\zeta) - \frac{1}{4\pi^2}\int_{\partial\Omega^+} \phi(\zeta)H(\zeta,z)\omega(\zeta) \tag{3.3}$$

for $z \in \Omega^+$, where

$$A(z) = \tilde{\Phi}(z) + \frac{1}{2\pi^2}\int_{\Omega^-} \tilde{\phi}(\zeta)\frac{d\sigma(\zeta)}{(1-\langle z,\zeta\rangle)^2}. \tag{3.4}$$

As noted above, since $\tilde{\Phi}(z) \in A(\mathbb{B})$, $\tilde{\Phi} \in CR(S)$, while the second term on the right-hand side of (3.4) is holomorphic in a neighborhood of Ω^+ in $\mathbb{C}^2$, and hence belongs to $CR(\Omega^+)$. This completes the proof. □

We refer to the formula appearing in Theorem 3.1 as the Cauchy-Green formula for Ω^+. Our goal in the remainder of this paper is to suggest how the formula of Theorem 3.1 can be used to derive approximation results for certain compact sets $K \subset S$, in much the same way that we employed the classical Cauchy-Green formula to derive the approximation results in section 1. We first note a simple corollary of Theorem 3.1; cf. the proof of the Hartogs-Rosenthal theorem in section 1.

Corollary 3.2. *Let K be a compact subset of S, and suppose for each $\epsilon > 0$, Ω_ϵ is a smoothly bounded domain in S containing K, with* $\lim_{\epsilon\to 0^+}\sigma(\Omega_\epsilon) = 0$. *For $\phi \in C^1(S)$ set*

$$F_\epsilon(z) = \frac{1}{4\pi^2}\int_{\partial\Omega_\epsilon} \phi(\zeta)H(\zeta,z)\,\omega(\zeta), \quad z \in \Omega_\epsilon.$$

If for each $\epsilon > 0$ there is a function $h_\epsilon \in CR(\Omega_\epsilon)$ so that $\lim_{\epsilon\to 0^+}\|F_\epsilon - h_\epsilon\|_K = 0$, *then $\phi|_K \in CR(K)$.*

Proof. Apply the formula of Theorem 3.1 to the domain Ω_ϵ to obtain for $z \in \Omega_\epsilon$,

$$\phi(z) = A(z) - h_\epsilon(z) - (F_\epsilon(z) - h_\epsilon(z)) + \frac{1}{4\pi^2}\int_{\Omega_\epsilon} \overline{\partial}\phi(\zeta)\wedge H(\zeta,z)\,\omega(\zeta) \tag{3.5}$$

where $A - h_\epsilon \in CR(\Omega_\epsilon)$. By hypothesis, $\|F_\epsilon - h_\epsilon\|_K$ tends to zero as $\epsilon \to 0^+$, while the uniform integrability of H in z implies that the last term on the right of (3.5) tends to zero uniformly in z as $\epsilon \to 0^+$ (see the proof of Lemma 4.1 below), and so

$$\lim_{\epsilon \to 0^+} \|\phi - (A + h_\epsilon)\|_K = 0,$$

implying $\phi \in CR(K)$. □

As Corollary 3.2 shows, the study of approximation by CR functions on compact subsets of S can be reduced to the study of CR approximation of integrals of the form

$$F_{\Omega,\phi}(z) = \int_{\partial\Omega} \phi(\zeta) H(\zeta, z)\, \omega(\zeta).$$

for smoothly bounded neighborhoods Ω of K. The remainder of this paper is devoted to an examination of such integrals. In section 4 we show that for certain Ω and ϕ, $F_{\Omega,\phi} \in CR(\Omega)$. In this case the formula of Theorem 3.1 immediately yields an estimate similar to (1.5). However, in general one cannot expect that F will be a CR function on Ω. In sections 5 and 6 we establish approximation results on certain compact subsets K of S with $\sigma(K) = 0$ by approximating $F_{\Omega_\epsilon,\phi}$ by CR functions on a sequence of domains Ω_ϵ shrinking to K, and applying Corollary 3.2.

Because the Cauchy-Green formula for domains on S lends itself to the study of approximation by CR functions, it is worthwhile to comment on the relationship between the space $CR(K)$ and the spaces $A(K), R(K)$ (consisting of uniform limits on K of functions holomorphic in neighborhood of K, and rational and holomorphic in a neighborhood of K, respectively.) Since any function holomorphic in a neighborhood U of K in $\mathbb{C}^2$ is a CR function on $U \cap S$, it follows that $A(K) \subset CR(K)$. On the other hand, if $\psi \in CR(\Omega)$ for some open neighborhood of K in S, it is well-known that there exists an open subset U of $\mathbb{B}$ containing Ω in its closure, and a function $\tilde{\psi}$ holomorphic in U and continuous on $\overline{U}$ with $\tilde{\psi} = \psi$ on Ω. There exists $t < 1$ so that $tz \in U$ for all $z \in K$, and so the function $z \mapsto \tilde{\psi}(tz)$ is holomorphic in a neighborhood of K in $\mathbb{C}^2$. As $t \to 1^-$, the functions $\tilde{\psi}(tz)$ approach ψ uniformly on K. It follows that $CR(K) \subset A(K)$, and hence $CR(K) = A(K)$. By the Stone-Weierstrass Theorem, $A(K) = C(K)$ if and only if the conjugate coordinate functions $\{\overline{z}_1, \overline{z}_2\}$ belong to $A(K)$; if (say) $z_2 \neq 0$ on $K \subset S$, then the relation $\overline{z}_2 = (1 - z_1\overline{z}_1)/z_2$ shows that $A(K) = C(K)$ if and only if $\overline{z}_1 \in A(K)$.

Recall that a compact subset K of $\mathbb{C}^n$ is said to be *rationally convex* if given any point $z \in \mathbb{C}^n \setminus K$, there is a polynomial P with $P(z) = 0$ but $P \neq 0$ on K. Rational convexity is a necessary, but far from sufficient, condition for $R(K) = C(K)$ when $n > 1$. Richard Basener [**2**] constructed rationally convex subsets K of S for which $R(K) \neq C(K)$. Basener's sets have positive σ-measure. We have been motivated by the following analogue of the Hartogs-Rosenthal Theorem, for which we have no proof or counterexample.

CONJECTURE 3.3. *Let K be a rationally convex subset of S with $\sigma(K) = 0$. Then $R(K) = C(K)$.*

If K is rationally convex, then it can be shown that $A(K) = R(K)$, so that by the above remarks, rational approximation on K is equivalent to CR approximation.

4. A special case

LEMMA 4.1. *Given a domain $\Omega^+ \subset S$, and $\phi \in C^1(\overline{\Omega^+})$, suppose that*

$$F(z) = \int_{\partial\Omega^+} \phi(\zeta) H(\zeta, z)\, \omega(\zeta)$$

defines a CR function on Ω^+. Then for any compact subset K of Ω^+,

$$\mathit{dist}(\phi, CR(K))) \equiv \inf\{\|\phi - \Psi\|_K : \Psi \in CR(K)\} \leq C_{\Omega^+} \cdot \|X(\phi)\|_{\overline{\Omega^+}} \tag{4.1}$$

for a positive constant C_{Ω^+} independent of ϕ and K with the property that for every $\epsilon > 0$, there exists $\delta > 0$ so that $\sigma(\Omega^+) < \delta \Longrightarrow C_{\Omega^+} < \epsilon$.

PROOF. By Theorem 3.1,

$$\phi(z) = A(z) + \frac{1}{4\pi^2} \int_{\Omega^+} \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\omega(\zeta) - \frac{1}{4\pi^2} \int_{\partial\Omega^+} \phi(\zeta) H(\zeta, z)\omega(\zeta) \tag{4.2}$$

with $A \in CR(\Omega^+)$. By hypothesis, the second integral on the right of (4.2) defines a function in $CR(S \setminus \partial\Omega^+)$ and so

$$\begin{aligned} \phi(z) &= A_1(z) + \frac{1}{4\pi^2} \int_{\Omega^+} \overline{\partial}\phi(\zeta) \wedge H(\zeta, z)\omega(\zeta) \\ &= A_1(z) + \frac{1}{2\pi^2} \int_{\Omega^+} X(\phi)(\zeta) H(\zeta, z) d\sigma(\zeta) \end{aligned}$$

with $A_1 \in CR(\Omega^+)$. Thus for any compact $K \subset \Omega^+$,

$$\mathrm{dist}(\phi, CR(K)) \leq \frac{1}{2\pi^2} \max_{\Omega^+} |X(\phi)| \int_{\Omega^+} |H(\zeta, z)|\, d\sigma(\zeta).$$

Set $C_{\Omega^+} = \sup_{z \in \Omega^+} (1/2\pi^2) \int_{\Omega^+} |H(\zeta, z)|\, d\sigma(\zeta)$. By the unitary invariance of H and $d\sigma$,

$$\int_{\Omega^+} |H(\zeta, z)|\, d\sigma(\zeta) = \int_{U_z(\Omega^+)} |H(\zeta, z^0)|\, d\sigma(\zeta) \leq \int_S |H(\zeta, z^0)| < \infty$$

where $z^0 = (1, 0)$ and U_z is a unitary transformation with $\det(U_z) = 1$ taking z to z^0. Thus C_{Ω^+} is finite and (4.1) holds. Moreover, since $H(\cdot, z^0) \in L^1(d\sigma)$, given $\epsilon > 0$ there exists $\delta > 0$ so that whenever Y is a measurable subset of S with $\sigma(Y) < \delta$, then $\int_Y |H(\zeta, z^0)| d\sigma(\zeta) < \epsilon$. The claim regarding C_{Ω^+} then follows by noting that for every z, $\sigma(\Omega^+) = \sigma(U_z \Omega^+)$. □

REMARK 4.2. The corresponding estimate for $\Omega = S$ (without hypothesis on ϕ) is given in [**1**], Theorem 4.1.

Next we identify a special case in which the integral F appearing in Lemma 4.1 can be shown to be a CR function on Ω^+.

LEMMA 4.3. *Let Ω^+ be a smoothly bounded domain on S and let $\phi \in C^1(\overline{\Omega^+})$. Suppose there exists a smooth 3-manifold-with-boundary Σ in $\mathbb{C}^2$ with $\partial\Sigma = \partial\Omega^+$, and a function $\tilde{\phi}$ smooth on Σ and holomorphic in a neighborhood of $\Sigma \setminus \partial\Sigma$, with $\phi = \tilde{\phi}$ on $\partial\Omega^+$. Then*

$$F(z) = \int_{\partial\Omega^+} \phi(\zeta) H(\zeta, z)\, \omega(\zeta)$$

belongs to $CR(S \setminus \partial\Omega^+)$.

PROOF. Let μ be the measure $\phi\omega\,|_{\partial\Omega^+}$. For any polynomial P, by Stokes' theorem we have

$$\int_S P d\mu = \int_{\partial\Omega^+} P\phi\,\omega = \int_\Sigma d(P\tilde{\phi}\,\omega) = \int_\Sigma \overline{\partial}(P\tilde{\phi}\,\omega) = 0$$

by assumption on ϕ. Therefore, $K_\mu \in CR(S \setminus \partial\Omega^+)$. But

$$K_\mu(z) = \int_S H(\zeta, z)\, d\mu(\zeta) = -F(z).$$

By property (iv) of K_μ above (section 2), $K_\mu \in CR(S \setminus \operatorname{supp}(\mu)) = CR(S \setminus \partial\Omega^+)$, so $F \in CR(S \setminus \partial\Omega^+)$. □

EXAMPLE 4.4. As a first example of the situation considered in Lemma 4.2, take

$$\Omega^+ = \{\zeta \in S : \operatorname{Im}(\zeta_2) > 0\},\ \Sigma = \{z \in \overline{\mathbb{B}} : \operatorname{Im}(\zeta_2) = 0\},\ \phi(\zeta) = \overline{\zeta}_2.$$

Then $\tilde{\phi}(\zeta) = \zeta_2$ extends ϕ from $\partial\Omega^+$ holomorphically to a neighborhood of Σ, and so

$$F(z) = \int_{\partial\Omega^+} \overline{\zeta}_2 H(\zeta, z)\,\omega(\zeta)$$

EXAMPLE 4.5. For $0 < r < 1$, let T_r be the torus

$$\{\zeta \in S : |\zeta_1| = r\}.$$

The three-manifold $\Sigma_r = \{\zeta \in \overline{\mathbb{B}} : |\zeta_1| = r\}$ has boundary T_r. Since for $\zeta \in T_r$

$$\overline{\zeta}_1 = \frac{r^2}{\zeta_1},\ \overline{\zeta}_2 = \frac{1 - r^2}{\zeta_2},$$

while $\zeta_1, \zeta_2 \neq 0$ near Σ_r, both $\overline{\zeta}_1$ and $\overline{\zeta}_2$ extend holomorphically from T_r to a neighborhood of Σ_r. By Lemma 4.3, (taking say $\Omega^+ = \{\zeta \in S : |\zeta_1| > r\}$), $\int_{T_r} \overline{\zeta}_j H(\zeta, z)\omega(\zeta)$ is a CR function on $S \setminus T_r$, j = 1,2. It follows that if $0 < a < b < 1$ and

$$\Omega = \{\zeta \in S : a < |\zeta_1| < b\},$$

then

$$\int_{\partial\Omega} \overline{\zeta}_j H(\zeta, z)\,\omega(\zeta) = \int_{T_b} \overline{\zeta}_j H(\zeta, z)\,\omega(\zeta) - \int_{T_a} \overline{\zeta}_j H(\zeta, z)\,\omega(\zeta)$$

defines a CR function on $S \setminus (T_a \cup T_b)$, $j = 1, 2$.

EXAMPLE 4.6. Let γ be a simple closed curve contained in the open unit disk of the complex plane, and let D be the region bounded by γ. Put

$$\Omega^+ = \{\zeta \in S : \zeta_1 \in D\}.$$

We will show that for any $\phi \in C(\gamma)$, $F(z) \in CR(S \setminus \partial\Omega^+)$, where

$$F(z) = \int_{\partial\Omega^+} \phi(\zeta_1) H(\zeta, z)\omega(\zeta).$$

For $\phi \in C(\gamma)$ and $\zeta \in \partial\Omega^+$, define $\tilde{\phi}(\zeta_1, \zeta_2) = \phi(\zeta_1)$. If Σ is the three-manifold

$$\Sigma = \{(\zeta_1, \zeta_2) : \zeta_1 \in \gamma, |\zeta_2| < \sqrt{1 - |\zeta_1|^2}\}$$

then $\partial\Sigma = \partial\Omega^+$. If $g \in C(\gamma)$ extends to be holomorphic in some neighborhood of γ, then $\tilde{g}$ extends to be holomorphic in a neighborhood of Σ, and so by Lemma 4.3

$$F_g(z) = \int_{\partial\Omega^+} \tilde{g}(\zeta) H(\zeta, z)\,\omega(\zeta)$$

defines a CR function on $S \setminus \partial\Omega^+$. But the functions holomorphic in a neighborhood of γ, restricted to γ, are dense in $C(\gamma)$, and so given any $\phi \in C(\gamma)$ we may choose a sequence $\{g_n\}$ of functions, each holomorphic in some neighborhood of γ, so that $\tilde{g}_n \to \tilde{\phi}$ uniformly on $\partial\Omega^+$. Then $F_{g_n} \to F$ uniformly on compact subsets of $S \setminus \partial\Omega^+$ and so $F \in CR(S \setminus \partial\Omega^+)$.

Note that by Lemma 4.1, each of the three preceding examples yields an approximation result for compact subsets of the respective domains Ω^+.

5. Certain two-spheres in S

For any $a, 0 < a < 1$, set $\lambda_a = \sqrt{1 - |a|^2}$ and let

$$S_a = \{(\zeta_1, \zeta_2) \in S : \mathrm{Im}(\zeta_2) = a$$

. Note that S_a is a two-sphere in the hyperplane $\{\mathrm{Im}(\zeta_2) = a\}$ defined by $|\zeta_1|^2 + \mathrm{Re}(\zeta_2)^2 = \lambda_a^2$. Fix $a_0, 0 < a_0 < 1$, and fix t with $0 < t < \lambda_{a_0}$. Henceforth we assume without comment that the parameter a is sufficiently close to a_0 so that also $t < \lambda_a$. Then $\triangle_a = \{\zeta \in S_a : |\mathrm{Re}(\zeta_2)| \geq t\}$ is a nonempty subset of S_a, consisting of two disjoint sets each diffeomorphic to a closed disk, that are neighborhoods of $(0, \pm\lambda_a + ia)$ in S_a. Set $M_a = S_a \setminus \triangle_a$. Our goal in this section is to use the formula of Theorem 3.1 to establish the following result, which can also be obtained by using results on approximation on totally real manifolds (see [**7**], [**8**], or [**9**]).

THEOREM 5.1. *If K is any compact subset of M_{a_0}, $A(K) = C(K)$.*

PROOF. To begin, we may parameterize M_a by

$$\zeta_1 = \sqrt{\lambda_a^2 - x^2}\, e^{i\theta},\ \zeta_2 = x + ia, \quad 0 \leq \theta \leq 2\pi,\ |x| < t. \tag{5.1}$$

Define

$$G_a(z) = \int_{M_a} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta), \quad z \in S \setminus M_a.$$

We may use properties (i) and (iii) of the Henkin kernel from section 1 to compute $X(G_a)$, obtaining

$$F_a(z) \equiv X(G_a)(z) = \int_{M_a} \frac{\overline{\zeta}_1 \omega(\zeta)}{(1 - \langle \zeta, z \rangle)^2}, \quad z \in S \setminus M_a. \tag{5.2}$$

Note that F_a is in fact defined for all $z \in \mathbb{B} \cup S \setminus M_a$, since for such z, $\langle z, \zeta \rangle \neq 1$ for $\zeta \in M_a$, and F_a is anti-holomorphic in $\mathbb{B}$.

LEMMA 5.2.

$$F_a(z) = 2\pi i \int_{-t}^{t} (x^2 - \lambda_a^2) \frac{dx}{(1 - (x + ia)\overline{z}_2)^2}, \quad z \in S \setminus M_a.$$

PROOF. Using the parametrization (5.1) we have

$$\omega = i\sqrt{\lambda_a^2 - x^2}\, e^{i\theta}\, d\theta \wedge dx$$

and so

$$F_a(z) = \int_{-t}^{t} (\lambda_a^2 - x^2) \left(\int_0^{2\pi} \frac{id\theta}{(A - Be^{i\theta})^2} \right) dx, \tag{5.3}$$

where $A = 1 - (x+ia)\overline{z}_2$ and $B = \sqrt{\lambda_a^2 - x^2}\,\overline{z}_1$. If $|z| < 1/2$, then $|B| < 1/2$ and $|A| > 1/2$, and thus $A - B\tau \neq 0$ for $|\tau| \leq 1$. The inner integral in (5.3) can be computed by the residue theorem:

$$\int_0^{2\pi} \frac{id\theta}{(A - Be^{i\theta})^2} = \int_{\tau=1} \frac{d\tau}{\tau} \frac{1}{(A - B\tau)^2} = \frac{2\pi i}{A^2}.$$

Thus

$$F_a(z) = 2\pi i \int_{-t}^{t} (x^2 - \lambda_a^2) \frac{dx}{(1 - (x+ia)\overline{z}_2)^2} \tag{5.4}$$

for $|z| < 1/2$. Since both $F_a(z)$ and the right-hand side of (5.4) are real-analytic in $\mathbb{B}$, the equality of (5.4) in fact holds for $z \in \mathbb{B}$. Moreover, since $|x+ia|^2 = x^2 + a^2 \leq t^2 + a^2 < \lambda_a^2 + a^2 = 1$, both sides of (5.4) are continuous on $\overline{\mathbb{B}} \setminus M_a$. It follows that the equality of (5.4) obtains for $z \in \overline{\mathbb{B}} \setminus M_a$. □

Define

$$I_a(z) = -\frac{2\pi i}{z_1} \int_{-t}^{t} \frac{(x^2 - \lambda_a^2)}{(x+ia)(1 - (x+ia)\overline{z}_2)}\, dx, \quad z \in S \setminus M_a,\ z_1 \neq 0.$$

Then I_a is smooth on $S \setminus M_a$ (note $|x + ia| \geq a > 0$), and a calculation gives

$$X(I_a) = F_a(z), \quad z \in S \setminus M_a$$

and therefore $X(I_a) = X(G_a)$ on $S \setminus M_a$. This yields:

Lemma 5.3.

$$\int_{M_a} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta) = I_a + \psi_a, \quad z \in S \setminus M_a$$

where $\psi_a \in CR(S \setminus M_a)$.

Now consider the domain Ω_ϵ, defined by

$$\Omega_\epsilon = \{\zeta \in S : a_0 - \epsilon < \mathrm{Im}(\zeta_2) < a_0 + \epsilon,\ |\mathrm{Re}(\zeta_2)| < t\}$$

where ϵ is chosen small enough so that $t < \lambda_a$ for all $a \in [a_0 - \epsilon, a_0 + \epsilon]$. Note

$$\partial\Omega_\epsilon = M_{a_0-\epsilon} \cup M_{a_0+\epsilon} \cup Y_t^\epsilon \cup Y_{-t}^\epsilon$$

with appropriate orientations, where

$$Y_{\pm t}^\epsilon = \{\zeta \in S : \mathrm{Re}(\zeta_2) = \pm t,\ a_0 - \epsilon < \mathrm{Im}(\zeta_2) < a_0 + \epsilon\},$$

and so by Lemma 5.3,

$$\begin{aligned}\int_{\partial\Omega_\epsilon} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta) &= h_\epsilon(z) + [I_{a_0+\epsilon}(z) - I_{a_0-\epsilon}(z)] \\ &+ \int_{Y_t^\epsilon} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta) - \int_{Y_{-t}^\epsilon} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta)\end{aligned}$$

for $z \in \Omega_\epsilon$, where $h_\epsilon \equiv \psi_{a_0-\epsilon} - \psi_{a_0+\epsilon} \in CR(\Omega_\epsilon)$. It is easy to check for $z \in M_{a_0}$ that $I_a(z)$ is a continuous function of a at a_0, so that

$$\lim_{\epsilon \to 0^+} I_{a_0+\epsilon}(z) - I_{a_0-\epsilon}(z) = 0, \quad z \in M_{a_0},$$

and in fact the limit is uniform on M_{a_0}. Moreover, restricting z to a compact subset K of M_{a_0}, the function $\zeta \mapsto \overline{\zeta}_1 H(\zeta, z)$ is bounded on $Y^{\epsilon}_{\pm t}$. Since the two-dimensional Hausdorff measure of Y_t approaches zero as $\epsilon \to 0$, and since ω is absolutely continuous with respect to Hausdorff measure, we see that

$$\lim_{\epsilon \to 0^+} \int_{Y^{\epsilon}_{\pm t}} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta) = 0$$

uniformly for $z \in K$. Combining these observations, we have

$$\lim_{\epsilon \to 0^+} \int_{\partial \Omega_\epsilon} \overline{\zeta}_1 H(\zeta, z)\omega(\zeta) - h_\epsilon(z) = 0$$

uniformly on compact subsets of M_{a_0}. By Corollary 3.2, since $h_\epsilon \in CR(\Omega_\epsilon)$, $\phi(\zeta) = \overline{\zeta}_1 \in CR(K)$. Since $\zeta_2 \neq 0$ on K, we conclude that (see the remarks at the end of section 3) $A(K) = C(K)$, and the proof of Theorem 5.1 is complete. ☐

6. Certain graphs in S

In this section we establish approximation on certain graphs in S, using the general method of section 5. Let D be a smoothly bounded plane domain with compact closure in the open disk $\mathbb{D}$, and suppose $f \in C^1(\overline{D})$ satisfies $|f(\zeta)| = \sqrt{1 - |\zeta|^2}$, so that the graph $\Gamma_f = \{(\zeta, f(\zeta)) : \zeta \in D\}$ lies in S. As in section 5, we will study the integral

$$\int_{\partial \Omega_\epsilon} \phi(\zeta) H(\zeta, z)\omega(\zeta)$$

on a sequence of domains Ω_ϵ for which $\Omega_\epsilon \downarrow \Gamma_f$ as $\epsilon \downarrow 0$. We begin with a representation for an integral of this type over Γ_f.

Lemma 6.1. *With f, D, Γ_f as above, and $\phi \in C^1(\Gamma_f)$, set*

$$G(z) = \int_{\Gamma_f} \phi(\zeta) H(\zeta, z)\omega(\zeta)$$

for $z \in S \setminus \Gamma_f$. Then (differentiation is in the z variable)

$$\overline{z}_2 X G(z) = -\int_{\partial D} \tilde{\phi}(\zeta_1) \frac{d\zeta_1}{1 - \zeta_1 \overline{z}_1 - f(\zeta_1)\overline{z}_2} + \int_D \frac{\partial \tilde{\phi}}{\partial \overline{\zeta}_1} \frac{d\overline{\zeta}_1 \wedge d\zeta_1}{1 - \zeta_1 \overline{z}_1 - f(\zeta_1)\overline{z}_2},$$

where $\tilde{\phi}(\zeta_1) = \phi(\zeta_1, f(\zeta_1))$.

Proof. Use properties (i) and (iii) of the Henkin kernel to write

$$\begin{aligned}
\overline{z}_2 X G(z) &= \overline{z}_2 \int_{\Gamma_f} \phi(\zeta)(1 - \langle \zeta, z\rangle)^{-2}\omega(\zeta) \\
&= \overline{z}_2 \int_D \tilde{\phi}(\zeta_1) \frac{d\zeta_1 \wedge (\partial f/\partial \overline{\zeta}_1) d\overline{\zeta}_1}{(1 - \zeta_1 \overline{z}_1 - f(\zeta_1)\overline{z}_2)^2} \\
&= \int_D \tilde{\phi}(\zeta_1) \frac{\partial}{\partial \overline{\zeta}_1} \left(\frac{1}{1 - \zeta_1 \overline{z}_1 - f(\zeta_1)\overline{z}_2} \right) d\zeta_1 \wedge d\overline{\zeta}_1
\end{aligned}$$

Rewrite the latter integral and use Stokes' Theorem to obtain

$$\begin{aligned}\overline{z}_2XG(z) &= -\int_D d\left(\frac{\tilde{\phi}(\zeta_1)d\zeta_1}{1-\zeta_1\overline{z}_1-f(\zeta_1)\overline{z}_2}\right)+\int_D\frac{\partial\tilde{\phi}}{\partial\overline{\zeta}_1}\cdot\frac{d\overline{\zeta}_1\wedge d\zeta_1}{1-\zeta_1\overline{z}_1-f(\zeta_1)\overline{z}_2}\\ &= -\int_{\partial D}\frac{\tilde{\phi}(\zeta_1)d\zeta_1}{1-\zeta_1\overline{z}_1-f(\zeta_1)\overline{z}_2}+\int_D\frac{\partial\tilde{\phi}}{\partial\overline{\zeta}_1}\cdot\frac{d\overline{\zeta}_1\wedge d\zeta_1}{1-\zeta_1\overline{z}_1-f(\zeta_1)\overline{z}_2}.\end{aligned}$$

□

We now restrict our attention to graphs of the following form. Fix r_0, r_1 with $0 < r_0 < r_1 < 1$ and let D be the annulus $\{\lambda \in \mathbb{C} : r_0 < |\lambda| < r_1\}$. For $\lambda \in D$ set

$$f(\lambda) = \sqrt{1-|\lambda|^2}\, B(\frac{\lambda}{|\lambda|}) \tag{6.1}$$

where B is a (fixed) finite Blaschke product:

$$B(\lambda) = \prod_{j=1}^{n}\frac{\lambda-\alpha_j}{1-\bar{\alpha}_j\lambda}$$

with $|\alpha_j| < 1, j = 1,\dots,n$. Denote by Γ_f the graph $\{(\lambda, f(\lambda)) : \lambda \in D\}$; note that since $|B(\tau)| = 1$ when $|\tau| = 1$, $\Gamma_f \subset S$.

THEOREM 6.2. *Assume f has the form (6.1). Then for any compact subset K of $\Gamma_f, A(K) = C(K)$.*

PROOF. For $t \in \mathbb{R}$ set $f_t(\lambda) = f(\lambda)e^{it}$. We consider first the integral

$$G_t = \int_{\Gamma_{f_t}}\overline{\zeta}_1 H(\zeta, z)\,\omega(\zeta)$$

LEMMA 6.3. *For $z \in S\setminus\Gamma_{f_t}$,*

$$\overline{z}_2XG_t(z) = 2\pi i\left(\frac{r_0^2}{1-b_t(r_0)\overline{z}_2}-\frac{r_1^2}{1-b_t(r_1)\overline{z}_2}\ +2\int_{r_0}^{r_1}\frac{rdr}{1-b_t(r)\overline{z}_2}\right)$$

where

$$b_t(r) = \sqrt{1-r^2}\cdot e^{it}B(0).$$

PROOF. According to Lemma 6.1,

$$\overline{z}_2XG_t(z) = -\int_{\partial D}\overline{\zeta}_1\frac{d\zeta_1}{1-\zeta_1\overline{z}_1-f_t(\zeta_1)\overline{z}_2}+\int_D\frac{d\overline{\zeta}_1\wedge d\zeta_1}{1-\zeta_1\overline{z}_1-f_t(\zeta_1)\overline{z}_2}. \tag{6.2}$$

Write the first integral on the right-hand side of (6.2) as $I_{r_1}(z) - I_{r_0}(z)$, where

$$\begin{aligned}I_r(z) &= i\int_0^{2\pi}\frac{r^2\,d\theta}{1-re^{i\theta}\overline{z}_1-\sqrt{1-r^2}B(e^{i\theta})e^{it}\overline{z}_2}\\ &= \int_{|\tau|=1}\frac{r^2\,d\tau}{\tau(1-r\tau\overline{z}_1-\sqrt{1-r^2}B(\tau)e^{it}\overline{z}_2)}.\end{aligned}$$

It is easy to check that $(1-r\tau\overline{z}_1-\sqrt{1-r^2}B(\tau)\overline{z}_2)^{-1}$ is holomorphic for $|\tau| \le 1$ and continuous for $|\tau| \le 1$ for fixed $z \in S\setminus\Gamma_{f_t}$. Cauchy's integral formula then gives

$$I_r(z) = \frac{2\pi i r^2}{1-\sqrt{1-r^2}B(0)e^{it}\overline{z}_2} = \frac{2\pi i r^2}{1-b_t(r)\overline{z}_2}. \tag{6.3}$$

The second integral on the right-hand side of (6.2) can be evaluated as follows: write $\zeta_1 = re^{i\theta}$, then

$$\begin{aligned}\int_D \frac{d\overline{\zeta}_1 \wedge d\zeta_1}{1-\zeta_1\overline{z}_1 - f_t(\zeta_1)\overline{z}_2} &= 2\int_{r_0}^{r_1}\left(\int_0^{2\pi}\frac{id\theta}{1-re^{i\theta}\overline{z}_1-\sqrt{1-r^2}B(e^{i\theta})e^{it}\overline{z}_2}\right) r\, dr \\ &= 2\int_{r_0}^{r_1}\left(\int_{|\tau|=1}\frac{d\tau}{\tau(1-r\tau\overline{z}_1-\sqrt{1-r^2}B(\tau)e^{it}\overline{z}_2}\right) r\, dr.\end{aligned}$$

It is easy to check that the function $(1-r\tau\overline{z}_1-\sqrt{1-r^2}B(\tau)e^{it}\overline{z}_2)$ is nonvanishing for $|\tau|\le 1$, if $z\in\mathbb{B}$ and so the Cauchy integral formula gives

$$\int_D \frac{d\overline{\zeta}_1 \wedge d\zeta_1}{1-\zeta_1\overline{z}_1 - f_t(\zeta_1)\overline{z}_2} = 2\int_{r_0}^{r_1}\frac{2\pi i}{1-\sqrt{1-r^2}B(0)e^{it}\overline{z}_2} r\, dr = 4\pi i\int_{r_0}^{r_1}\frac{r\, dr}{1-b_t(r)\overline{z}_2}.$$

Again by continuity this result holds also for $z\in S\setminus\Gamma_{f_t}$. Combining (6.2), (6.3)and the last displayed equation gives the formula of Lemma 6.3. ☐

We next proceed to construct a certain function M_t such that $XM_t = XG_t$ on $S\setminus\Gamma_t$. By Lemma 6.3,

$$\begin{aligned}\frac{1}{2\pi i}XG_t(z) &= \frac{1}{\overline{z}_2}\frac{r_0^2}{1-b_t(r_0)\overline{z}_2} - \frac{1}{\overline{z}_2}\frac{r_1^2}{1-b_t(r_1)\overline{z}_2} + 2\int_{r_0}^{r_1}\frac{r\, dr}{\overline{z}_2(1-b_t(r)\overline{z}_2)} \\ &= r_0^2\left(\frac{1}{\overline{z}_2}+\frac{b_t(r_0)}{1-b_t(r_0)\overline{z}_2}\right) - r_1^2\left(\frac{1}{\overline{z}_2}+\frac{b_t(r_1)}{1-b_t(r_1)\overline{z}_2}\right) \\ &\quad -2\int_{r_0}^{r_1}\left(\frac{1}{\overline{z}_2}+\frac{b_t(r)}{1-b_t(r)\overline{z}_2}\right) r\, dr.\end{aligned}$$

and so

$$\frac{1}{2\pi i}XG_t(z) = \frac{r_0^2 b_t(r_0)}{1-b_t(r_0)\overline{z}_2} - \frac{r_1^2 b_t(r_1)}{1-b_t(r_1)\overline{z}_2} + 2\int_{r_0}^{r_1}\frac{b_t(r)}{1-b_t(r)\overline{z}_2} r\, dr.$$

Note that

$$X\left[\frac{1}{z_1}\log(1-b_t(r)\overline{z}_2)\right] = \frac{b_t(r)}{1-b_t(r)\overline{z}_2},$$

and so if we set

$$M_t(z) = \frac{2\pi i}{z_1}\left(r_0^2\log(1-b_t(r_0)\overline{z}_2) - r_1^2\log(1-b_t(r_1)\overline{z}_2) + 2\int_{r_0}^{r_1}\log(1-b_t(r)\overline{z}_2)\, rdr\right)$$

we have

$$XM_t = XG_t \tag{6.4}$$

on $S\setminus\Gamma_{f_t}$.

Given a compact subset K of Γ_f, for fixed $\epsilon > 0$ we define a neighborhood Ω_ϵ of K in S by

$$\Omega_\epsilon = \{(\zeta_1,\zeta_2) : \zeta_1\in D, \zeta_2 = f(\zeta_1)e^{it}, |t|<\epsilon\}.$$

Note that $\{\Omega_\epsilon\}_{\epsilon>0}$ forms a decreasing family of domains with intersection Γ_f, and that

$$\partial\Omega_\epsilon = \Gamma_{f_\epsilon}\cup\Gamma_{f_{-\epsilon}}\cup B_\epsilon$$

where

$$B_\epsilon = \{(\lambda, f(\lambda)e^{it}) : \lambda\in\partial D, |t|<\epsilon\}.$$

Set

$$F_\epsilon(z) = \int_{\partial\Omega_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta).$$

Then for $z \in \Omega_\epsilon$,

$$\begin{aligned} F_\epsilon(z) &= \int_{\Gamma_{f_\epsilon}} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta) - \int_{\Gamma_{f_{-\epsilon}}} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta) + \int_{B_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta) \\ &= G_\epsilon(z) - G_{-\epsilon}(z) + \int_{B_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta) \\ &= M_\epsilon(z) - M_{-\epsilon}(z) + h_\epsilon(z) + \int_{B_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta), \end{aligned}$$

where $h_\epsilon = G_\epsilon - M_\epsilon - (G_{-\epsilon} - M_{-\epsilon}) \in CR(\Omega_\epsilon)$, by (6.4), and so

$$F_\epsilon(z) - h_\epsilon(z) = M_\epsilon(z) - M_{-\epsilon}(z) + \int_{B_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta). \tag{6.5}$$

Since the two-dimensional measure of B_ϵ tends to zero with epsilon it follows that (cf. the end of section 5)

$$\lim_{\epsilon\to 0^+} \int_{B_\epsilon} \overline{\zeta}_1 H(\zeta,z)\omega(\zeta) = 0$$

uniformly on K. Moreover, an examination of the definition of M_t shows that $\lim_{\epsilon\to 0^+} M_\epsilon - M_{-\epsilon} = 0$ uniformly on K. It follows by (6.5) that

$$\lim_{\epsilon\to 0^+} \|F_\epsilon - h_\epsilon\|_K = 0.$$

By Corollary 3.2, $\phi(\zeta) = \overline{\zeta}_1 \in CR(K)$. Since $\zeta_2 \neq 0$ on K, we conclude that (see the remarks at the end of section 3) $A(K) = C(K)$, and the proof of Theorem 6.2 is complete. □

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