Boundary Behavior of some Cauchy Transforms

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Abstract

Let Ω be a relatively open subset of the unit sphere $\partial\mathbb{B}_n = \{z \in \mathbb{C}^n : \|z\| = 1\}$, with smooth boundary Γ relative to $\partial\mathbb{B}_n$, and let χ_Ω be the characteristic function of Ω. Let $E(\Gamma)$ denote the set of points $z \in \Gamma$ such that the manifold Γ is not generic at z. We show that if $f \in C^1(\partial\mathbb{B}_n)$ the Cauchy transform of $f\chi_\Omega$ extends continuously to each point of $\partial\mathbb{B}_n \setminus E(\Gamma)$.

§1. Introduction

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, $\partial\mathbb{B}_n$ its boundary, and σ the standard invariant $2n-1$ dimensional measure on $\partial\mathbb{B}_n$, normalized so that $\sigma(\mathbb{B}_n) = 1$. For $f \in L^1(d\sigma)$, the Cauchy transform of f is a function holomorphic in $\mathbb{B}_n$, defined by

$$\mathcal{C}[f](z) := \int_{\partial\mathbb{B}_n} f(\zeta)C(z,\zeta)d\sigma(\zeta), \quad z \in \mathbb{B}_n$$

where

$$C(z,\zeta) := \frac{1}{(1-\langle z,\zeta\rangle)^n}$$

and $\langle z,\zeta\rangle := \sum_{j=1}^n z_j\overline{\zeta}_j$ is the standard Hermitian inner product. Under mild smoothness assumptions on f (say if f satisfies a Hölder condition with exponent α, $0 < \alpha < 1$ on $\partial\mathbb{B}_n$ - see [2], Theorems 6.4.9 and 6.4.10) $\mathcal{C}[f]$ is known to extend continuously to the closed ball. Also, if f is the restriction to $\partial\mathbb{B}_n$ of a function F in the ball algebra $A(\mathbb{B}_n)$ (the space of functions holomorphic on $\mathbb{B}_n$ and continuous on its closure), then $\mathcal{C}[f] = F$, and so $\mathcal{C}[f]$ extends to be continuous on the closed ball.

In this paper we consider the boundary behavior of Cauchy transforms of certain bounded discontinuous functions. Suppose Ω is a relatively open subset of $\partial\mathbb{B}_n$ with smooth boundary (relative to $\partial\mathbb{B}_n$) Γ, let χ_Ω be the characteristic function of Ω, and suppose $f \in C^1(\mathbb{B}_n)$. If $z^0 \in \partial\mathbb{B}_n \setminus \Omega$, it is clear that $\mathcal{C}[f\chi_\Omega]$ extends holomorphically to a neighborhood of z^0, since the Cauchy kernel $C(z,\zeta)$ is holomorphic as a function of z near z^0 for each $\zeta \in \Omega$. The fact that $\mathcal{C}[f] = \mathcal{C}[f\chi_{\partial\mathbb{B}_n\setminus\Omega}] + \mathcal{C}[f\chi_\Omega]$ then implies that $\mathcal{C}[\chi_\Omega](z)$ extends continuously to $\overline{\mathbb{B}_n} \setminus \Gamma$.

When $n = 1$, this is all that can be said: if Ω is the arc $\{e^{i\theta} : \alpha < \theta < \beta\}$ for some fixed α, β, and $f \equiv 1$, then

$$\mathcal{C}[\chi_\Omega](z) = \frac{1}{2\pi}\int_\alpha^\beta \frac{1}{1-ze^{-i\theta}}\,d\theta = \frac{1}{2\pi}\left[i\log\left(\frac{1-ze^{-i\alpha}}{1-ze^{-i\beta}}\right) + \beta - \alpha\right], \quad |z| < 1.$$

Thus $\mathcal{C}[\chi_\Omega](z)$ is unbounded as z approaches $\{e^{i\alpha}, e^{i\beta}\} = \Gamma$. However, when $n > 1$, $\mathcal{C}[\chi_\Omega](z)$ may possess limits as $z \to z_0$ for certain points $z_0 \in \Gamma$. Consider

$$\Omega = \{(\zeta_1,\zeta_2) \in \partial\mathbb{B}_2 : \Re\zeta_1 > 0\},$$

whose boundary is the two-sphere

$$\Gamma = \{(\zeta_1, \zeta_2) : \Re\zeta_1 = 0, (\Im\zeta_1)^2 + |z_2|^2 = 1\}$$

A computation reveals that $\mathcal{C}[\chi_\Omega](z_1, z_2)$ is a function of z_1 alone, that

$$\mathcal{C}[\chi_\Omega](z_1) = \frac{i}{\pi} \cdot \frac{1 - z_1^2}{z_1^2} \cdot \log\left(\frac{1 - iz_1}{1 + iz_1}\right) + \frac{1}{\pi z_1} + \frac{1}{2}, \quad |z_1| < 1, z_1 \neq 0 \tag{1}$$

and that the function defined by the right-hand side of (1) has a removable singularity at $z_1 = 0$. It is apparent that $\mathcal{C}[\chi_\Omega](z)$ possesses a limit as $z \in \mathbb{B}_2$ approaches every point of $\partial\mathbb{B}_2$ except $(\pm i, 0)$. These are precisely the two points in Γ where the tangent space to Γ is a complex line.

It turns out that the phenomenon exhibited in the last example is a general one. If Γ is a smooth real submanifold of $\mathbb{C}^n$, we let $E(\Gamma)$ be the set of points $p \in \Gamma$ where the tangent space to Γ is generic (see section 2 for the definition).

Theorem 1. *Let Ω be a relatively open subset of $\partial\mathbb{B}_n$ whose boundary Γ (relative to $\partial\mathbb{B}_n$) is a submanifold of $\partial\mathbb{B}_n$ of class C^2. If $f \in C^1(\partial\mathbb{B}_n)$, then $\mathcal{C}[f\chi_\Omega]$ extends continuously to $\partial\mathbb{B}_2 \setminus E(\Gamma)$.*

In section 2 we prove (with Γ as in Theorem 1) that the $2n-2$-dimensional Hausdorff measure of $E(\Gamma)$ is zero. The proof of Theorem 1 is given in section 3. Note that the set Ω in the theorem need not be connected, so that the result holds for finite unions of smoothly bounded connected domains on the sphere with disjoint closures. In section 4 we give an example of a compact set $Y \subset \partial\mathbb{B}_2$ with empty interior and positive σ-measure such that such that $\mathcal{C}[\chi_Y]$ extends continuously to the closed ball.

The first author encountered integrals of the form $\mathcal{C}[f\chi_\Omega]$ in working on questions of rational approximation on subsets of $\partial\mathbb{B}_2$ with John Wermer, whom he thanks for enlightening discussions.

§2 The measure of $E(\Gamma)$.

Let M be a smooth real k-dimensional submanifold of $\mathbb{C}^n$. Assume $k \geq n$. Near a given $p \in M$, M may be defined as the common zero set of $2n-k$ smooth real-valued functions $\rho_1, \ldots, \rho_{2n-k}$ with $d\rho_1, \ldots, d\rho_{2n-k}$ linearly independent over $\mathbb{R}$. Let $r(p)$ be the rank (over $\mathbb{C}$) of the matrix $((\partial\rho_i(p)/\partial\overline{z}_j))$. Then the dimension (over $\mathbb{C}$) of the maximal complex subspace $\mathcal{T}_pM$ of the real tangent space T_pM is $n - r(p)$. When this rank is maximal (i.e., $r(p) = 2n-k$, in which case $\mathcal{T}_pM$ has dimension $k-n$), we say that M is generic at p. Let $E(M)$ be the set of non-generic points of M.

Lemma 2.1 *Let $M = \Gamma$ be a C^2-smooth $2n-2$ dimensional submanifold of a strictly pseudoconvex hypersurface Σ in $\mathbb{C}^n$. Then the $2n-2$-dimensional Hausdorff measure of $E(\Gamma)$ is zero.*

Proof. The proof is virtually identical to that given in [AI] for the case $n = 2$. Fix $p \in \Gamma$. After a linear change of coordinates in $\mathbb{C}^n$ we may choose a neighborhood U of p so that

$$\Gamma \cap U = \{(z', h(z') : z' \in U'\}$$

where $z' = (z_1, \dots, z_{n-1})$, U' is a neighborhood of the origin in $\mathbb{C}^{n-1}$, and h is a smooth complex-valued function on U'. Set

$$X = \{z' \in U' : \frac{\partial h}{\partial \overline{z}_j}(z') = 0, \forall j = 1, \dots, n-1\}$$

Using the defining functions $\Re[z_n - h(z')], \Im[z_n - h(z')]$ for Γ we see that

$$E(\Gamma) \cap U = \{(z', h(z')) : z' \in X\}.$$

If we show that X has $2n-2$ dimensional Lebesgue measure zero (as a subset of $\mathbb{R}^{2n-2}$), then $E(\Gamma) \cap U$ has $2n-2$ Hausdorff measure zero. Since p was arbitrary, Γ can be then be covered by countably many sets of $2n-2$ Hausdorff measure zero, and the proof will be complete.

Assume to the contrary that X has positive $2n-2$ Lebesgue measure. Lemma 3.2 of [AI] implies that

$$\frac{\partial^2 h}{\partial \overline{z}_j \partial z_k}(z') = 0, \forall j, k = 1, \dots, n-1 \tag{2}$$

at almost all points z' in X. We may therefore assume that (2) holds at the origin $0'$ in $\mathbb{C}^{n-1}$ and that $h(0') = 0$. Since $\mathbf{0} := (0', 0) \in E(\Gamma)$, $\dim_{\mathbb{C}} \mathcal{T}_{\mathbf{0}}\Gamma = n-1$, and $\mathcal{T}_{\mathbf{0}}\Gamma = \mathcal{T}_{\mathbf{0}}\Sigma$. By another linear transformation of $\mathbb{C}^n$ we may assume

$$\mathcal{T}_{\mathbf{0}}\Sigma = \{(z', 0) : z' \in \mathbb{C}^{n-1}\} \tag{3}$$

and thus

$$h(0') = 0 \text{ and } \frac{\partial h}{\partial z_j}(0') = \frac{\partial h}{\partial \overline{z}_j}(0') = 0, \; j = 1, \dots, n-1. \tag{4}$$

By the strict pseudoconvexity of Σ, we may choose a defining function ρ for Σ near $\mathbf{0}$ so that $\rho(\mathbf{0}) = 0, d\rho(\mathbf{0}) \neq 0$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \overline{z}_j \partial z_k}(\mathbf{0}) a_j \overline{a}_k > 0$$

whenever $\sum_{j=1}^{n} a_j \partial\rho/\partial\overline{z}_j(\mathbf{0}) = 0$. In particular, since (3) implies $\partial\rho/\partial\overline{z}_1(\mathbf{0}) = 0$, we must have

$$\frac{\partial^2 \rho}{\partial \overline{z}_1 \partial z_1}(\mathbf{0}) > 0. \tag{5}$$

However, if we differentiate the identity

$$\rho(z', h(z')) \equiv 0$$

with respect to z_1 and then $\overline{z}_1$, and use (4) and (2), we find $\partial^2\rho/\partial\overline{z}_1\partial z_1(\mathbf{0}) = 0$, contradicting (5). This completes the proof. □

§3 Proof of Theorem 1

The standard Euclidean norm in $\mathbb{C}^n$ will be denoted by $\|z\|$, and the ball of radius r and center z will be denoted by $B(z, r)$. We make use of the following elementary lemma, whose proof we omit.

Lemma 3.1 *Suppose $g \in C^1(\mathbb{B}_n)$, $z^0 \in \partial\mathbb{B}_n$, and that there exist a neighborhood N of z^0 in $\partial\mathbb{B}_n$, $0 < t^* < 1$, and a function $\psi \in L^1([t^*, 1])$ such that*

$$\left|\frac{\partial}{\partial t} g(tz)\right| \leq \psi(t)$$

for all $z \in N$ and all $t, t^* < t < 1$. Then $g^*(z) := \lim_{t\to 1^-} g(tz)$ exists for all $z \in N$, and the extension of g defined by

$$\tilde{g}(\zeta) := \begin{cases} g(z) & z \in \mathbb{B}_n, \\ g^*(z) & z \in N \end{cases}$$

is continuous at z^0.

Now let Ω, Γ, and f be as in Theorem 1. By the remarks in the introduction, $\mathcal{C}[f\chi_\Omega]$ extends continuously to each point of $\mathbb{B}_n \setminus \Gamma$. Therefore to prove Theorem 1 it suffices to show that $\mathcal{C}[f\chi_\Omega]$ extends to be continuous at z^0 for each $z^0 \in \Gamma \setminus E(\Gamma)$. To do this we will apply Lemma 3.1.

Fix $z^0 \in \Gamma \setminus E(\Gamma)$. We claim that there exist a neighborhood U of z^0 in $\mathbb{B}_n$ and a vector field

$$L_\zeta = \sum_{j=1}^{n} \alpha_j \frac{\partial}{\partial \bar{\zeta}_j} + \beta_j \frac{\partial}{\partial \zeta_j}$$

with coefficients $\alpha_j, \beta_j \in C^1(U)$ so that

$$L_\zeta(C(tz,\zeta)) = t\frac{\partial}{\partial t} C(tz,\zeta), \forall\, \zeta, z, t, \tag{6}$$

$$L_\zeta \in T_\zeta \partial\mathbb{B}_n \otimes \mathbb{C}, \quad \forall \zeta \in U \tag{7}$$

and

$$L_\zeta \in T_\zeta \Gamma \otimes \mathbb{C}, \quad \forall \zeta \in U \cap \Gamma. \tag{8}$$

In fact, (6) holds with $\alpha_j = \bar{\zeta}_j$, by a routine computation. Let $\rho_1(\zeta) = \sum_{j=1}^n |\zeta_j|^2 - 1$ be the defining function for $\mathbb{B}_n$. We may choose a neighborhood U of z^0 and a real-valued function $\rho_2 \in C^2(U)$ such that

$$\Omega \cap U = \{\zeta \in U : \rho_2(\zeta) < 0\}$$

and $d\rho_1 \wedge d\rho_2 \neq 0$ on U. Then (7) and (8) will be satisfied provided

$$L_\zeta(\rho_1) = L_\zeta(\rho_2) \tag{9}$$

for all $\zeta \in U$. The condition that Γ is generic at z^0 implies that (shrinking U if necessary) $\partial\rho_1 \wedge \partial\rho_2 \neq 0$ on U , which in turn implies that the system (9) may be solved for the coefficients $\beta_j \in C^1(U)$ and establishes the claim.

Now assume that the neighborhood U and L_ζ are chosen to satisfy (6), (7) and (8). Choose $\eta > 0$ so that $B(z^0, 2\eta) \subset U$, and choose $\phi \in C^\infty(\partial\mathbb{B}_n)$ such that $\phi \equiv 1$ on $B(z^0, \eta/2)$ and $\phi \equiv 0$ on $\partial\mathbb{B}_n \setminus B(z^0, \eta)$. For $0 < t < 1$,

$$\frac{\partial}{\partial t}\mathcal{C}[f\chi_\Omega](tz) = \int_\Omega f(\zeta)\frac{\partial}{\partial t} C(tz,\zeta)\, d\sigma(\zeta) = I_1(tz) + I_2(tz)$$

where

$$I_1(tz) = \int_\Omega (1-\phi(\zeta))f(\zeta)\frac{\partial}{\partial t}C(tz,\zeta)\,d\sigma(\zeta) = \int_{\Omega\setminus B(z^0,\eta/2)} (1-\phi(\zeta))f(\zeta)\frac{\partial}{\partial t}C(tz,\zeta)\,d\sigma(\zeta).$$

and

$$I_2(tz) = \int_\Omega \phi(\zeta)f(\zeta)\frac{\partial}{\partial t}C(tz,\zeta)\,d\sigma(\zeta) = \frac{1}{t}\int_{\Omega\cap B(z^0,2\eta)} \phi(\zeta)f(\zeta)L_\zeta C(tz,\zeta)\,d\sigma(\zeta).$$

Let $N = B(z^0,\eta/4)$. The inner product $\langle tz,\zeta\rangle$ is bounded away from 1 for $z\in N$, $\zeta\in\Omega\setminus B(z^0,\eta/2)$ and $t<1$, and so there exists a constant K_1 so that

$$|I_1(tz)| \le K_1 \tag{10}$$

for all $t<1, \zeta\in N$. To estimate I_2 we integrate by parts. Using properties (ii) and (iii) of L_ζ and the fact that the integrand vanishes outside $B(z^0,\eta)$, we obtain

$$I_2(tz) = \frac{1}{t}\int_{\Omega\cap B(z^0,2\eta)} h(\zeta)C(tz,\zeta)\,d\sigma(\zeta)$$

for some continuous function h vanishing outside $B(z^0,\eta)$. Thus

$$|I_2(tz)| \le \frac{\|h\|_\infty}{t}\int_{\partial\mathbb{B}_n} |C(tz,\zeta)|\,d\sigma(\zeta).$$

The unitary invariance of C and $d\sigma$ implies that the latter integral is independent of z, and so if $e_1 = (1,0,\dots,0)$, we have

$$\int_{\partial\mathbb{B}_n} |C(tz,\zeta)|\,d\sigma(\zeta) = \int_{\partial\mathbb{B}_n} |C(te_1,\zeta)|\,d\sigma(\zeta) = \int_{\partial\mathbb{B}_n} |1-t\bar\zeta_1|^{-n}\,d\sigma(\zeta)$$

Using formula 1.4.5 (2) of [2],

$$\int_{\partial\mathbb{B}_n} |1-t\bar\zeta_1|^{-n}\,d\sigma(\zeta) = \frac{n-1}{\pi}\int_{\mathbb{D}} \frac{(1-|\lambda|^2)^{n-2}}{|1-t\bar\lambda|^n}\,dm(\lambda)$$

where $\mathbb{D}$ is the unit disk in the plane and m is normalized Lebesgue measure on $\mathbb{D}$. Let $\mathbb{D}^+ = \{\lambda\in\mathbb{D} : \Re\lambda > 0\}$. Since

$$\int_{\mathbb{D}\setminus\mathbb{D}^+} \frac{(1-|\lambda|^2)^{n-2}}{|1-t\bar\lambda|^n}\,dm(\lambda)$$

is bounded independently of $t, 0<t<1$, it suffices to estimate

$$\int_{\mathbb{D}^+} \frac{(1-|\lambda|^2)^{n-2}}{|1-t\bar\lambda|^n}\,dm(\lambda).$$

It is easy to check that if $1/2<t<1$, $|1-\bar\lambda|^2 \le 4|1-t\bar\lambda|^2$ for $\lambda\in\mathbb{D}^+$, and therefore

$$\frac{(1-|\lambda|^2)^{n-2}}{|1-t\bar\lambda|^n} \le \frac{(1-|\lambda|)^{n-2}}{|1-t\bar\lambda|^n} \le \frac{|1-\bar\lambda|^{n-2}}{|1-t\bar\lambda|^{n-2}}\cdot\frac{1}{|1-t\bar\lambda|^2} \le 2^{n-2}\frac{1}{|1-t\bar\lambda|^2}$$

It follows that, for $1/2 < t < 1$,

$$\int_{\mathbb{D}^+} \frac{(1-|\lambda|^2)^{n-2}}{|1-t\bar{\lambda}|^n}\, dm(\lambda) \le 2^{n-2} \int_{\mathbb{D}^+} \frac{1}{|1-t\bar{\lambda}|^2}\, dm(\lambda) \le 2^{n-2} \int_{\mathbb{D}} \frac{1}{|1-t\bar{\lambda}|^2}\, dm(\lambda).$$

An explicit integration shows that the latter integral is $O(|\log(1-t)|)$ as $t \to 1^-$. Therefore there exist constants K_2, K_3 so that

$$|I_2(tz)| \le K_2 + K_3 |\log(1-t)| \tag{11}$$

for all t, $1/2 < t < 1$ and $\zeta \in N$. Combining (10) and (11) we see that the hypotheses of Lemma 3.1 hold for $g = \mathcal{C}[f\chi_\Omega]$, $t^* = 1/2$, and $\psi(t) = K_1 + K_2 + K_3|\log(1-t)|$. This completes the proof of Theorem 1. □

§4 An Example.

If Γ is generic at each point $p \in \Gamma$, Theorem 1 implies that $\mathcal{C}[f\chi_\Omega]$ extends continuously to the closed ball when f is smooth. Consider, for example,

$$\Omega := \{\zeta = (\zeta_1, \zeta_2) \in \partial\mathbb{B}_n : \zeta_1 \in D(a,r)\}$$

where $D(a,r)$ is the disk $\{\lambda \in \mathbb{C} : |\lambda - a| = r\}$, with a, r chosen so that $\overline{D(a,r)} \subset \mathbb{D}$. Then $\Gamma = \{\zeta = (\zeta_1,\zeta_2) \in \partial\mathbb{B}_n : |\zeta_1 - a| = r\}$ is a totally real torus. Take $f \equiv 1$. A computation gives

$$\mathcal{C}[\chi_\Omega](z_1, z_2) = \frac{r^2}{(1-az_1)^2} \tag{12}$$

Now choose disks as in the construction of a "Swiss cheese": fix $R < 1$, let $D_0 = D(0,R)$, and choose a_j, r_j so that if $D_j := D(a_j, r_j)$ then (i) $\overline{D_j} \subset D_0$; (ii) $D_j \cap D_k = \emptyset$ if $j \ne k$; (iii) $\sum_{j=1}^{\infty} r_j < \infty$ and (iv) $X := \overline{D_0} \setminus \cup_{j=1}^{\infty} D_j$ has empty interior. It is known that then X has positive two-dimensional measure. If

$$Y := \{\zeta = (\zeta_1, \zeta_2) \in \partial\mathbb{B}_n : \zeta_1 \in X\}$$

then Y is a compact set with empty interior and $\sigma(Y) > 0$. Using (12) we see that

$$\mathcal{C}[\chi_Y](z_1, z_2) = R^2 - \sum_{j=1}^{\infty} \frac{r_j^2}{(1-a_jz_1)^2},$$

the sum converging uniformly on the closed ball.

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