Boundary Behavior of some Cauchy Transforms

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Abstract

Let $\Omega$ be a relatively open subset of the unit sphere $\partial \mathbb{B}_n = \{ z \in \mathbb{C}^n : \|z\| = 1 \}$, with smooth boundary $\Gamma$ relative to $\partial \mathbb{B}_n$, and let $\chi_\Omega$ be the characteristic function of $\Omega$. Let $E(\Gamma)$ denote the set of points $z \in \Gamma$ such that the manifold $\Gamma$ is not generic at $z$. We show that if $f \in C^1(\partial \mathbb{B}_n)$ the Cauchy transform of $f \chi_\Omega$ extends continuously to each point of $\partial \mathbb{B}_n \setminus E(\Gamma)$.

§1. Introduction

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$, $\partial \mathbb{B}_n$ its boundary, and $\sigma$ the standard invariant $2n - 1$ dimensional measure on $\partial \mathbb{B}_n$, normalized so that $\sigma(\partial \mathbb{B}_n) = 1$. For $f \in L^1(\partial \mathbb{B}_n)$, the Cauchy transform of $f$ is a function holomorphic in $\mathbb{B}_n$, defined by

$$C[f](z) := \int_{\partial \mathbb{B}_n} f(\zeta)C(z, \zeta) d\sigma(\zeta), \quad z \in \mathbb{B}_n$$

where

$$C(z, \zeta) := \frac{1}{(1 - \langle z, \zeta \rangle)^n}$$

and $\langle z, \zeta \rangle := \sum_{j=1}^{n} z_j \overline{\zeta}_j$ is the standard Hermitian inner product. Under mild smoothness assumptions on $f$ (say if $f$ satisfies a Hölder condition with exponent $\alpha$, $0 < \alpha < 1$ on $\partial \mathbb{B}_n$ - see [2], Theorems 6.4.9 and 6.4.10) $C[f]$ is known to extend continuously to the closed ball. Also, if $f$ is the restriction to $\partial \mathbb{B}_n$ of a function $F$ in the ball algebra $A(\mathbb{B}_n)$ (the space of functions holomorphic on $\mathbb{B}_n$ and continuous on its closure), then $C[f] = F$, and so $C[f]$ extends to be continuous on the closed ball.

In this paper we consider the boundary behavior of Cauchy transforms of certain bounded discontinuous functions. Suppose $\Omega$ is a relatively open subset of $\partial \mathbb{B}_n$ with smooth boundary (relative to $\partial \mathbb{B}_n$) $\Gamma$, let $\chi_\Omega$ be the characteristic function of $\Omega$, and suppose $f \in C^1(\partial \mathbb{B}_n)$. If $z^0 \in \partial \mathbb{B}_n \setminus \Omega$, it is clear that $C[f \chi_\Omega]$ extends holomorphically to a neighborhood of $z^0$, since the Cauchy kernel $C(z, \zeta)$ is holomorphic as a function of $z$ near $z^0$ for each $\zeta \in \Omega$. The fact that $C[f] = C[f \chi_{\partial \mathbb{B}_n \setminus \Omega}] + C[f \chi_{\partial \mathbb{B}_n}]$ then implies that $C[\chi_\Omega](z)$ extends continuously to $\mathbb{B}_n \setminus \Gamma$.

When $n = 1$, this is all that can be said: if $\Omega$ is the arc $\{ e^{i\theta} : \alpha < \theta < \beta \}$ for some fixed $\alpha, \beta$, and $f \equiv 1$, then

$$C[\chi_\Omega](z) = \frac{1}{2\pi} \int_{\alpha}^{\beta} \frac{1}{1 - ze^{-i\theta}} d\theta = \frac{1}{2\pi} \left[ i \log \left( \frac{1 - ze^{-i\alpha}}{1 - ze^{-i\beta}} \right) + \beta - \alpha \right], \quad |z| < 1.$$  

Thus $C[\chi_\Omega](z)$ is unbounded as $z$ approaches $\{ e^{i\alpha}, e^{i\beta} \} = \Gamma$. However, when $n > 1$, $C[\chi_\Omega](z)$ may possess limits as $z \to z_0$ for certain points $z_0 \in \Gamma$. Consider

$$\Omega = \{ (\zeta_1, \zeta_2) \in \partial \mathbb{B}_2 : \Re \zeta_1 > 0 \},$$

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whose boundary is the two-sphere
\[ \Gamma = \{ (\zeta_1, \zeta_2) : \Re \zeta_1 = 0, (3\zeta_1)^2 + |z_2|^2 = 1 \} \]
A computation reveals that \( \mathcal{C}[\chi_{\Omega}](z_1, z_2) \) is a function of \( z_1 \) alone, that
\[
\mathcal{C}[\chi_{\Omega}](z_1) = \frac{i}{\pi} \cdot \frac{1 - z_2^2}{z_1^2} \cdot \log \left( \frac{1 - iz_1}{1 + iz_1} \right) + \frac{1}{2pi} + \frac{1}{2}, |z_1| < 1, z_1 \neq 0
\]
and that the function defined by the right-hand side of (1) has a removable singularity at \( z_1 = 0 \). It is apparent that \( \mathcal{C}[\chi_{\Omega}](z) \) possesses a limit as \( z \in \mathbb{B}_2 \) approaches every point of \( \partial \mathbb{B}_2 \) except \((\pm i, 0)\). These are precisely the two points in \( \Gamma \) where the tangent space to \( \Gamma \) is a complex line.

It turns out that the phenomenon exhibited in the last example is a general one. If \( \Gamma \) is a smooth real submanifold of \( \mathbb{C}^n \), we let \( E(\Gamma) \) be the set of points \( p \in \Gamma \) where the tangent space to \( \Gamma \) is generic (see section 2 for the definition).

**Theorem 1.** Let \( \Omega \) be a relatively open subset of \( \partial \mathbb{B}_n \) whose boundary \( \Gamma \) (relative to \( \partial \mathbb{B}_n \)) is a submanifold of \( \partial \mathbb{B}_n \) of class \( C^2 \). If \( f \in C^1(\partial \mathbb{B}_n) \), then \( \mathcal{C}[f\chi_{\Omega}] \) extends continuously to \( \partial \mathbb{B}_2 \setminus E(\Gamma) \).

In section 2 we prove (with \( \Gamma \) as in Theorem 1) that the \( 2n - 2 \)-dimensional Hausdorff measure of \( E(\Gamma) \) is zero. The proof of Theorem 1 is given in section 3. Note that the set \( \Omega \) in the theorem need not be connected, so that the result holds for finite unions of smoothly bounded connected domains on the sphere with disjoint closures. In section 4 we give an example of a compact set \( Y \subset \partial \mathbb{B}_2 \) with empty interior and positive \( \sigma \)-measure such that such that \( \mathcal{C}[\chi_{Y}] \) extends continuously to the closed ball.

The first author encountered integrals of the form \( \mathcal{C}[f\chi_{\Omega}] \) in working on questions of rational approximation on subsets of \( \partial \mathbb{B}_2 \) with John Wermer, whom he thanks for enlightening discussions.

**§2 The measure of \( E(\Gamma) \).**

Let \( M \) be a smooth real \( k \)-dimensional submanifold of \( \mathbb{C}^n \). Assume \( k \geq n \). Near a given \( p \in M \), \( M \) may be defined as the common zero set of \( 2n - k \) smooth real-valued functions \( \rho_1, \ldots, \rho_{2n-k} \) with \( dp_1, \ldots, dp_{2n-k} \) linearly independent over \( \mathbb{R} \). Let \( r(p) \) be the rank (over \( \mathbb{C} \) of the matrix \( ((\partial \rho_i(p)/\partial \zeta_j)) \). Then the dimension (over \( \mathbb{C} \) of the maximal complex subspace \( \mathcal{T}_p M \) of the real tangent space \( T_p M \) is \( n - r(p) \). When this rank is maximal (i.e., \( r(p) = 2n - k \), in which case \( \mathcal{T}_p M \) has dimension \( k - n \)), we say that \( M \) is generic at \( p \). Let \( E(M) \) be the set of non-generic points of \( M \).

**Lemma 2.1** Let \( M = \Gamma \) be a \( C^2 \)-smooth \( 2n - 2 \)-dimensional submanifold of a strictly pseudoconvex hypersurface \( \Sigma \) in \( \mathbb{C}^n \). Then the \( 2n - 2 \)-dimensional Hausdorff measure of \( E(\Gamma) \) is zero.

**Proof.** The proof is virtually identical to that given in [AI] for the case \( n = 2 \). Fix \( p \in \Gamma \). After a linear change of coordinates in \( \mathbb{C}^n \) we may choose a neighborhood \( U \) of \( p \) so that
\[
\Gamma \cap U = \{ (z', h(z') : z' \in U' \}
\]
where \( z' = (z_1, \ldots, z_{n-1}) \), \( U' \) is a neighborhood of the origin in \( \mathbb{C}^{n-1} \), and \( h \) is a smooth complex-valued function on \( U' \). Set
\[
X = \{ z' \in U' : \frac{\partial h}{\partial z_j}(z') = 0, \forall j = 1, \ldots, n-1 \}
\]
Using the defining functions \( \Re[z_n - h(z')], \Im[z_n - h(z')] \) for \( \Gamma \) we see that
\[
E(\Gamma) \cap U = \{ (z', h(z')) : z' \in X \}.
\]
If we show that \( X \) has \( 2n - 2 \) dimensional Lebesgue measure zero (as a subset of \( \mathbb{R}^{2n-2} \)), then \( E(\Gamma) \cap U \) has \( 2n - 2 \) Hausdorff measure zero. Since \( p \) was arbitrary, \( \Gamma \) can be then be covered by countably many sets of \( 2n - 2 \) Hausdorff measure zero, and the proof will be complete.

Assume to the contrary that \( X \) has positive \( 2n - 2 \) Lebesgue measure. Lemma 3.2 of [Al] implies that
\[
\frac{\partial^2 h}{\partial z_j \partial z_k}(z') = 0, \forall j, k = 1, \ldots, n-1 \tag{2}
\]
at almost all points \( z' \) in \( X \). We may therefore assume that \( (2) \) holds at the origin \( 0' \) in \( \mathbb{C}^{n-1} \) and that \( h(0') = 0 \). Since \( 0 := (0', 0) \in E(\Gamma) \), \( \dim_{\mathbb{C}} T_0 \Gamma = n - 1 \), and \( T_0 \Gamma = T_0 \Sigma \). By another linear transformation of \( \mathbb{C}^n \) we may assume
\[
T_0 \Sigma = \{ (z', 0) : z' \in \mathbb{C}^{n-1} \} \tag{3}
\]
and thus
\[
h(0') = 0 \quad \text{and} \quad \frac{\partial h}{\partial z_j}(0') = \frac{\partial h}{\partial z_j}(0') = 0, j = 1, \ldots, n-1. \tag{4}
\]
By the strict pseudoconvexity of \( \Sigma \), we may choose a defining function \( \rho \) for \( \Sigma \) near \( 0 \) so that \( \rho(0) = 0, d\rho(0) \neq 0 \) and
\[
\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial z_k}(0) a_j \bar{a}_k > 0
\]
whenever \( \sum_{j=1}^n a_j \partial \rho / \partial z_j(0) = 0 \). In particular, since \( (3) \) implies \( \partial \rho / \partial \bar{z}_1(0) = 0 \), we must have
\[
\frac{\partial^2 \rho}{\partial \bar{z}_1 \partial z_1}(0) > 0. \tag{5}
\]
However, if we differentiate the identity
\[
\rho(z', h(z')) \equiv 0
\]
with respect to \( z_1 \) and then \( \bar{z}_1 \), and use \( (4) \) and \( (2) \), we find \( \partial^2 \rho / \partial \bar{z}_1 \partial z_1(0) = 0 \), contradicting \( (5) \). This completes the proof. \( \square \)

§3 Proof of Theorem 1

The standard Euclidean norm in \( \mathbb{C}^n \) will be denoted by \( \| z \| \), and the ball of radius \( r \) and center \( z \) will be denoted by \( B(z, r) \). We make use of the following elementary lemma, whose proof we omit.
Lemma 3.1 Suppose \( g \in C^1(\mathbb{B}_n) \), \( z^0 \in \partial \mathbb{B}_n \), and that there exist a neighborhood \( N \) of \( z^0 \) in \( \partial \mathbb{B}_n \), \( 0 < t^* < 1 \), and a function \( \psi \in L^1([t^*, 1]) \) such that

\[
\left| \frac{\partial}{\partial t} g(tz) \right| \leq \psi(t)
\]

for all \( z \in N \) and all \( t, t^* < t < 1 \). Then \( g^*(z) := \lim_{t \to 1^-} g(tz) \) exists for all \( z \in N \), and the extension of \( g \) defined by

\[
\bar{g}(\zeta) := \begin{cases} 
  g(z) & z \in \mathbb{B}_n, \\
  g^*(z) & z \in N 
\end{cases}
\]

is continuous at \( z^0 \).

Now let \( \Omega, \Gamma, \) and \( f \) be as in Theorem 1. By the remarks in the introduction, \( C[f\chi_{\Omega}] \) extends continuously to each point of \( \mathbb{B}_n \setminus \Gamma \). Therefore to prove Theorem 1 it suffices to show that \( C[f\chi_{\Omega}] \) extends to be continuous at \( z^0 \) for each \( z^0 \in \Gamma \setminus E(\Gamma) \). To do this we will apply Lemma 3.1.

Fix \( z^0 \in \Gamma \setminus E(\Gamma) \). We claim that there exist a neighborhood \( U \) of \( z^0 \) in \( \mathbb{B}_n \) and a vector field

\[
L_\zeta = \sum_{j=1}^n \alpha_j \frac{\partial}{\partial \zeta_j} + \beta_j \frac{\partial}{\partial \zeta_j}
\]

with coefficients \( \alpha_j, \beta_j \in C^1(U) \) so that

\[
L_\zeta(C(tz, \zeta)) = t \frac{\partial}{\partial t} C(tz, \zeta), \forall \zeta, z, t,
\]

\[
L_\zeta \in T_\zeta \partial \mathbb{B}_n \otimes \mathbb{C}, \forall \zeta \in U
\]

and

\[
L_\zeta \in T_\zeta \Gamma \otimes \mathbb{C}, \forall \zeta \in U \cap \Gamma.
\]

In fact, (6) holds with \( \alpha_j = \bar{\zeta}_j \), by a routine computation. Let \( \rho_1(\zeta) = \sum_{j=1}^n |\zeta_j|^2 - 1 \) be the defining function for \( \mathbb{B}_n \). We may choose a neighborhood \( U \) of \( z^0 \) and a real-valued function \( \rho_2 \in C^2(U) \) such that

\[
\Omega \cap U = \{ \zeta \in U : \rho_2(\zeta) < 0 \}
\]

and \( d\rho_1 \wedge d\rho_2 \neq 0 \) on \( U \). Then (7) and (8) will be satisfied provided

\[
L_\zeta(\rho_1) = L_\zeta(\rho_2)
\]

for all \( \zeta \in U \). The condition that \( \Gamma \) is generic at \( z^0 \) implies that (shrinking \( U \) if necessary) \( \partial \rho_1 \wedge \partial \rho_2 \neq 0 \) on \( U \), which in turn implies that the system (9) may be solved for the coefficients \( \beta_j \in C^1(U) \) and establishes the claim.

Now assume that the neighborhood \( U \) and \( L_\zeta \) are chosen to satisfy (6), (7) and (8). Choose \( \eta > 0 \) so that \( B(z^0, 2\eta) \subset U \), and choose \( \phi \in C^\infty(\partial \mathbb{B}_n) \) such that \( \phi \equiv 1 \) on \( B(z^0, \eta/2) \) and \( \phi \equiv 0 \) on \( \partial \mathbb{B}_n \setminus B(z^0, \eta) \). For \( 0 < t < 1 \),

\[
\frac{\partial}{\partial t} C[f\chi_{\Omega}](tz) = \int_\Omega f(\zeta) \frac{\partial}{\partial t} C(tz, \zeta) \, d\sigma(\zeta) = I_1(tz) + I_2(tz)
\]
where
\[ I_1(tz) = \int_{\Omega} (1 - \phi(\zeta)) f(\zeta) \frac{\partial}{\partial t} C(tz, \zeta) \, d\sigma(\zeta) = \int_{\Omega \setminus B(z^0, \eta/2)} (1 - \phi(\zeta)) f(\zeta) \frac{\partial}{\partial t} C(tz, \zeta) \, d\sigma(\zeta). \]

and
\[ I_2(tz) = \int_{\Omega} \phi(\zeta) f(\zeta) \frac{\partial}{\partial t} C(tz, \zeta) \, d\sigma(\zeta) = \frac{1}{t} \int_{\Omega \cap B(z^0, 2\eta)} \phi(\zeta) f(\zeta) L_{\zeta} C(tz, \zeta) \, d\sigma(\zeta). \]

Let \( N = B(z^0, \eta/4) \). The inner product \( \langle tz, \zeta \rangle \) is bounded away from 1 for \( z \in N, \zeta \in \Omega \setminus B(z^0, \eta/2) \) and \( t < 1 \), and so there exists a constant \( K_1 \) so that
\[ |I_1(tz)| \leq K_1 \] (10)
for all \( t < 1, \zeta \in N \). To estimate \( I_2 \) we integrate by parts. Using properties (ii) and (iii) of \( L_{\zeta} \) and the fact that the integrand vanishes outside \( B(z^0, \eta) \), we obtain
\[ I_2(tz) = \frac{1}{t} \int_{\Omega \cap B(z^0, 2\eta)} h(\zeta) C(tz, \zeta) \, d\sigma(\zeta) \]
for some continuous function \( h \) vanishing outside \( B(z^0, \eta) \). Thus
\[ |I_2(tz)| \leq \frac{\|h\|_{\infty}}{t} \int_{\partial B_n} |C(tz, \zeta)| \, d\sigma(\zeta). \]

The unitary invariance of \( C \) and \( d\sigma \) implies that the latter integral is independent of \( z \), and so if \( e_1 = (1, 0, \ldots, 0) \), we have
\[ \int_{\partial B_n} |C(tz, \zeta)| \, d\sigma(\zeta) = \int_{\partial B_n} |C(te_1, \zeta)| \, d\sigma(\zeta) = \int_{\partial B_n} |1 - t\zeta_1|^{-n} \, d\sigma(\zeta) \]
Using formula 1.4.5 (2) of [2],
\[ \int_{\partial B_n} |1 - t\zeta_1|^{-n} \, d\sigma(\zeta) = \frac{n - 1}{\pi} \int_{\mathbb{D}} \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \, dm(\lambda) \]
where \( \mathbb{D} \) is the unit disk in the plane and \( m \) is normalized Lebesgue measure on \( \mathbb{D} \). Let \( \mathbb{D}^+ = \{ \lambda \in \mathbb{D} : \Re \lambda > 0 \} \). Since
\[ \int_{\mathbb{D} \setminus \mathbb{D}^+} \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \, dm(\lambda) \]
is bounded independently of \( t, 0 < t < 1 \), it suffices to estimate
\[ \int_{\mathbb{D}^+} \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \, dm(\lambda). \]
It is easy to check that if \( 1/2 < t < 1 \), \( |1 - \lambda|^2 \leq 4|1 - t\lambda|^2 \) for \( \lambda \in \mathbb{D}^+ \), and therefore
\[ \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \leq \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \leq \frac{|1 - \lambda|^n}{|1 - t\lambda|^n} \frac{1}{|1 - t\lambda|^2} \leq 2^{n-2} \frac{1}{|1 - t\lambda|^2} \]
It follows that, for $1/2 < t < 1$,
\[
\int_{D^+} \frac{(1 - |\lambda|^2)^{n-2}}{|1 - t\lambda|^n} \, dm(\lambda) \leq 2^{n-2} \int_{D^+} \frac{1}{|1 - t\lambda|^2} \, dm(\lambda) \leq 2^{n-2} \int_{\mathbb{D}} \frac{1}{|1 - t\lambda|^2} \, dm(\lambda).
\]

An explicit integration shows that the latter integral is $O(\log(1 - t))$ as $t \to 1^-$. Therefore there exist constants $K_2, K_3$ so that
\[
|J_2(tz)| \leq K_2 + K_3 \log(1 - t) \tag{11}
\]
for all $t, 1/2 < t < 1$ and $\zeta \in N$. Combining (10) and (11) we see that the hypotheses of Lemma 3.1 hold for $g = C[f\chi_2]$, $t^* = 1/2$, and $\psi(t) = K_1 + K_2 + K_3 \log(1 - t)$. This completes the proof of Theorem 1. $\Box$

§4 An Example.

If $\Gamma$ is generic at each point $p \in \Gamma$, Theorem 1 implies that $\mathcal{C}[f\chi_2]$ extends continuously to the closed ball when $f$ is smooth. Consider, for example,
\[
\Omega := \{ \zeta = (\zeta_1, \zeta_2) \in \partial \mathbb{B}_n : \zeta_1 \in D(a, r) \}
\]
where $D(a, r)$ is the disk $\{ \lambda \in \mathbb{C} : |\lambda - a| = r \}$, with $a, r$ chosen so that $\overline{D(a, r)} \subset \mathbb{D}$. Then $\Gamma = \{ \zeta = (\zeta_1, \zeta_2) \in \partial \mathbb{B}_n : |\zeta_1 - a| = r \}$ is a totally real torus. Take $f \equiv 1$. A computation gives
\[
\mathcal{C}[\chi_2](z_1, z_2) = \frac{r^2}{(1 - az_1)^2} \tag{12}
\]

Now choose disks as in the construction of a “Swiss cheese”: fix $R < 1$, let $D_0 = D(0, R)$, and choose $a_j, r_j$ so that if $D_j := D(a_j, r_j)$ then (i) $\overline{D_j} \subset D_0$; (ii) $D_j \cap D_k = \emptyset$ if $j \neq k$; (iii) $\sum_{j=1}^{\infty} r_j < \infty$ and (iv) $X := \overline{D_0} \setminus \bigcup_{j=1}^{\infty} D_j$ has empty interior. It is known that then $X$ has positive two-dimensional measure. If
\[
Y := \{ \zeta = (\zeta_1, \zeta_2) \in \partial \mathbb{B}_n : \zeta_1 \in X \}
\]
then $Y$ is a compact set with empty interior and $\sigma(Y) > 0$. Using (12) we see that
\[
\mathcal{C}[\chi_Y](z_1, z_2) = R^2 - \sum_{j=1}^{\infty} \frac{r_j^2}{(1 - a_j z_1)^2},
\]
the sum converging uniformly on the closed ball.

References


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