UNIFORM APPROXIMATION BY $\Box_b$–HARMONIC FUNCTIONS

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Abstract. The Mergelyan and Ahlfors-Beurling estimates for the Cauchy transform give quantitative information on uniform approximation by rational functions with poles off K. We will present an analogous result for an integral transform on the unit sphere in $\mathbb{C}^2$ introduced by Henkin, and show how it can be used to study approximation by functions that are locally harmonic with respect to the Kohn Laplacian $\Box_b$.

1. Introduction

The primary tool in the study of rational approximation on compact subsets of the plane is the Cauchy transform $\hat{\mu}$ of a compactly supported, complex Borel measure μ, defined by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}.$$

The following facts about $\hat{\mu}$ can be found in many sources (see, for example [3], [7], [8], [17]): $\hat{\mu}$ is finite a.e. with respect to Lebesgue measure m on the plane, vanishes at ∞, and satisfies

$$\partial \hat{u}/\partial \overline{z} = -\pi\mu \tag{1.1}$$

in the sense of distributions. If μ is absolutely continuous with respect to Lebesgue measure m on the plane, then $\hat{\mu}$ is continuous on $\mathbb{C}$ and thus is bounded. Mergelyan ([16], see also [9], or [7], Lemma 3.1.1) proved the following estimate:

$$\int_K \frac{dm(\zeta)}{|\zeta - z|} \leq 2\sqrt{\pi \cdot m(K)}. \tag{1.2}$$

This can be used to give a quantitative estimate for rational approximation as follows. For a compact set K, $C(K)$ will denote the set of all continuous functions on K with uniform norm $\|f\|_K = \max\{|f(z)| : z \in K\}$, and $R(K)$ will be the closure in $C(K)$ of the set of rational functions holomorphic in a neighborhood (allowed to depend on the function) of K. Let ϕ be any smooth compactly supported function on the plane. The Cauchy-Green formula (which also proves (1.1)) gives

$$\begin{aligned}\phi(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{\zeta}} \cdot \frac{1}{\zeta - z}\, dm(\zeta) \\ &= \frac{1}{\pi}\int_{\mathbb{C}\setminus K} \frac{\partial \phi}{\partial \overline{\zeta}} \cdot \frac{1}{\zeta - z}\, dm(\zeta) + \frac{1}{\pi}\int_{K} \frac{\partial \phi}{\partial \overline{\zeta}} \cdot \frac{1}{\zeta - z}\, dm(\zeta).\end{aligned} \tag{1.3}$$

Date: October 22, 2010.
Key words and phrases. Henkin kernel, Uniform Approximation, Kohn Laplacian.
2010 *Mathematics Subject Classification*: Primary 32V20; Secondary 32A26.

As a function of $z \in K$, the integral over the complement of K in (1.3) is easily seen to belong to $R(K)$, and so Mergelyan's estimate readily implies that

$$\operatorname{dist}(\phi, R(K)) := \inf\{\|\phi - g\|_K : g \in R(K)\} \le C\|\partial\phi/\partial\overline{z}\|_K \cdot \sqrt{m(K)} \tag{1.4}$$

with $C = 2/\sqrt{\pi}$. In particular, (1.4) gives an easy proof of the Hartogs-Rosenthal theorem: $m(K) = 0$ implies $R(K) = C(K)$.

An estimate similar to (1.2), due to Ahlfors and Beurling ([1], see also [3], [9]), states that

$$\left|\int_K \frac{dm(\zeta)}{\zeta - z}\right| \le \sqrt{\pi \cdot m(K)} \tag{1.5}$$

for all $z \in \mathbb{C}$. This can be used to give a more precise estimate for the "analytic content" $\lambda(K) := \operatorname{dist}(\overline{z}, R(K))$. Taking ϕ to be compactly supported and equal to $\overline{z}$ near K, (1.5) together with the Ahlfors-Beurling estimate gives

$$\lambda(K) \le \sqrt{m(K)/\pi}. \tag{1.6}$$

This inequality was first observed by H. Alexander ([2]). D. Khavinson ([13], see also [9]) established a lower bound for $\lambda(K)$ when K is a set of finite perimeter, in terms of the area and perimeter of K, and combining this with (1.6) gave a new proof of the isoperimetric inequality in the plane. In subsequent work ([14]), Khavinson gave geometric estimates for harmonic approximation in $\mathbb{R}^n$. We will say more about Khavinson's results in section 3.

Given a smoothly bounded strictly convex domain $\Omega \subset \mathbb{C}^n$, G. Henkin ([11]) constructed a kernel and transform on $\partial\Omega$ that bears some similarity to the Cauchy transform. In the case when Ω is the open unit ball $\mathbb{B}$ in $\mathbb{C}^2$, Henkin's kernel is defined for $\zeta \ne z$ in $S = \partial\mathbb{B}$ by

$$H(\zeta, z) = \frac{\langle Tz, \zeta\rangle}{|1 - \langle \zeta, z\rangle|^2},$$

where $\langle z, \zeta\rangle$ is the Hermitian inner product $\langle z, \zeta\rangle = z_1\overline{\zeta}_1 + z_2\overline{\zeta}_2$ and T is the transformation $Tz = (\overline{z}_2, -\overline{z}_1)$. (See [18] for information on Henkin's kernel on the sphere, and [4], [5], [6], [15] for applications to approximation theory.) It is clear that $H(Uz, U\zeta) = H(z, \zeta)$ for any unitary transformation U. For fixed $z \in S$, $H(z, \cdot)$ is integrable with respect to the standard invariant three-dimensional measure σ on S (uniformly in z, by the unitary invariance). Given a measure μ on S, define the Henkin transform K_μ of μ by

$$K_\mu(z) = \int_S H(\zeta, z)\, d\mu(\zeta), \quad z \in S.$$

Then (cf. equation (1.1))

$$L(K_\mu) = -2\pi^2\mu$$

in the sense of distributions, where L is the standard tangential Cauchy-Riemann operator on S, $L = z_2\partial/\partial\overline{z}_1 - z_1\partial/\partial\overline{z}_2$, provided that μ satisfies the (necessary) condition $\int P\, d\mu = 0$ for all holomorphic polynomials P. There is also an analogue of the Cauchy-Green formula involving H (see [5]):

$$\phi(z) = \Phi(z) + 2\int_S H(\zeta, z)L(\phi)(\zeta)\, d\sigma(\zeta), \tag{1.7}$$

where Φ belongs to the ball algebra $A(\mathbb{B})$ consisting of functions holomorphic in $\mathbb{B}$ and continuous on its closure. The equation (1.7) is well-suited for studying approximation by functions g satisfying $\overline{L}L(g) = 0$, where $\overline{L} = \overline{z}_2\partial/\partial z_1 - \overline{z}_1\partial/\partial z_2$ is the conjugate operator to L, for the following reason: a computation gives (we write L_z to indicate differentiation in the z-variable)

$$L_z(H(\zeta, z)) = (1 - \langle\zeta, z\rangle)^{-2} := C(\zeta, z),\ z \neq \zeta \in S, \tag{1.8}$$

where C is the Poisson-Szegö kernel on the sphere, and so for $\zeta \neq z \in S$

$$\overline{L_z}L_z H(\zeta, z) = 0, \tag{1.9}$$

since $C(\zeta, z)$ is anti-holomorphic in z. We note that as an operator on functions, the Kohn Laplacian $\Box_b$ on the sphere S is (up to a constant) equal to $\overline{L}L$ (see for example [10]), but we do not use this fact in any essential way.

If $U \subset S$ is (relatively) open, we say g is $\Box_b$–harmonic in U if $\overline{L}Lg = 0$ in U (in the weak sense). If $K \subset S$ is compact, we denote by $\mathcal{H}(K)$ the uniform closure in $C(K)$ of functions that are $\Box_b$–harmonic in a neighborhood of K, and for $f \in C(K)$ we set

$$\operatorname{dist}(f, \mathcal{H}(K)) := \inf\{\|f - g\|_K : g \in \mathcal{H}(K)\}.$$

If K has non-empty interior, then $f \in \mathcal{H}(K)$ is easily to seen to imply $\overline{L}L(f) = 0$ on $\operatorname{int}(K)$, and hence $\mathcal{H}(K) \neq C(K)$ if K has non-empty interior.

We will first establish an analogue of (1.2) for the Henkin transform:

Theorem 1.1. *There exists a constant $C > 0$ such that if $K \subset S$ is compact, then*

$$\int_K |H(z, \zeta)|\, d\sigma(\zeta) \leq C\sigma(K)^{1/4}$$

for all $z \in S$. Moreover, the exponent $1/4$ is the best possible, i.e., it cannot be replaced by any larger exponent.

This will allow us to easily conclude the following analogue of (1.4):

Theorem 1.2. *There exists $C > 0$ such that for all $\phi \in C^\infty(S)$ and all $K \subset S$ compact,*

$$\operatorname{dist}(\phi, \mathcal{H}(K)) \leq C\|L\phi\|_K \cdot \sigma(K)^{1/4}.$$

An immediate consequence of Theorem 1.2 is a Hartogs-Rosenthal-type result for uniform approximation by $\Box_b$–harmonic functions.

Corollary 1. If $\sigma(K) = 0$, then $\mathcal{H}(K) = C(K)$.

Proof. If $f \in C(K)$, we may choose a sequence $\phi_n \in C^\infty(S)$ with $\lim_{n\to\infty} \|f - \phi_n\|_K = 0$. By Proposition 1, if $\sigma(K) = 0$, $\phi_n \in \mathcal{H}(K)$ for each n, and so $f \in \mathcal{H}(K)$. □

In section 2 we give the proof of Theorem 1.1. Section 3 contains the brief proof of Theorem 1.2 as well as several remarks and open questions.

2. Proof of Theorem 1.1

We will make use of the formula (see for example [19], Lemma 1.10)

$$\int_S f d\sigma = \int_{\mathbb{D}} \int_0^{2\pi} f(\lambda, \sqrt{1 - |\lambda|^2}\, e^{i\theta})\, d\theta\, dm(\lambda), \tag{2.1}$$

for any $f \in L^1(d\sigma)$, where $\mathbb{D}$ is the unit disk in the complex plane and m is Lebesgue measure on the plane. (Neither m nor σ are normalized). Given quantities A, B depending on one or more variables we use the notation $A \lesssim B$ to indicate the existence of a positive constant k independent of the variables such that $A \le kB$, and $A \approx B$ to indicate that both $A \lesssim B$ and $B \lesssim A$ hold. We note that it suffices to prove the estimate of Theorem 1.1 when $\sigma(K)$ is sufficiently small, a fact we will use implicitly in what follows.

Let K be a compact subset of S. For fixed $z \in K$,

$$\left|\int_K H(\zeta, z)\, d\sigma(\zeta)\right| \le \int_K |H(\zeta, z)|\, d\sigma(\zeta) = \int_{K'} |H(\zeta, e_1)| d\sigma(\zeta), \tag{2.2}$$

where U is a unitary transformation with $U(z) = (1, 0) := e_1$, using the unitary invariance of H and σ, and $K' = \{U(\zeta) : \zeta \in K\}$. Since $\sigma(K') = \sigma(K)$, the estimate of Theorem 1.1 will be established if we can show that there exists $C > 0$ such that for all compact K,

$$\int_K |H(\zeta, e_1)| d\sigma(\zeta) \le C\sigma(K)^{1/4}. \tag{2.3}$$

Note that

$$\int_K |H(e_1, \zeta)| d\sigma(\zeta) = \int_K \frac{|\zeta_2|}{|1 - \zeta_1|^2}\, d\sigma(\zeta) = \int_K \frac{\sqrt{1 - |\zeta_1|^2}}{|1 - \zeta_1|^2}\, d\sigma(\zeta) = \int_K \tilde{h}\, d\sigma \tag{2.4}$$

where for $\lambda \in \mathbb{D}$,

$$h(\lambda) := \frac{\sqrt{1 - |\lambda|^2}}{|1 - \lambda|^2} \tag{2.5}$$

and $\tilde{h}(\zeta_1, \zeta_2) = h(\zeta_1)$. Let

$$\mathcal{D}_t := \{\lambda \in \mathbb{D} : h(\lambda) \ge t\}$$

and set

$$\widetilde{\mathcal{D}}_t = \{(\zeta_1, \zeta_2) \in S : \zeta_1 \in \mathcal{D}_t\} = \{\zeta \in S : \tilde{h}(\zeta) \ge t\}.$$

We now imitate the key step in the proofs of both the Mergelyan and Ahlfors-Beurling estimates. Since $\sigma(\widetilde{\mathcal{D}}_t)$ clearly depends continuously on t, we may choose t so that $\sigma(K) = \sigma(\widetilde{\mathcal{D}}_t)$. The equalities

$$\sigma(\widetilde{\mathcal{D}}_t) = \sigma(\widetilde{\mathcal{D}}_t \cap K) + \sigma(\widetilde{\mathcal{D}}_t \backslash K) \text{ and } \sigma(K) = \sigma(K \cap \widetilde{\mathcal{D}}_t) + \sigma(K \backslash \widetilde{\mathcal{D}}_t)$$

imply that $\sigma(K\backslash\widetilde{\mathcal{D}}_t) = \sigma(\widetilde{\mathcal{D}}_t\backslash K)$. Moreover, $\tilde{h} < t$ on $K\backslash\widetilde{\mathcal{D}}_t$ while $\tilde{h} \ge t$ on $\widetilde{\mathcal{D}}_t\backslash K$, so that

$$\int_{K\backslash\widetilde{\mathcal{D}}_t} \tilde{h}\, d\sigma < t\sigma(K \setminus \widetilde{\mathcal{D}}_t) = t\sigma(\widetilde{\mathcal{D}}_t \setminus K) \le \int_{\widetilde{\mathcal{D}}_t\backslash K} \tilde{h}\, d\sigma.$$

Therefore,

$$\begin{aligned} \int_K \tilde{h}\, d\sigma &= \int_{K\cap\widetilde{\mathcal{D}}_t} \tilde{h}\, d\sigma + \int_{K\backslash\widetilde{\mathcal{D}}_t} \tilde{h}\, d\sigma \\ &< \int_{K\cap\widetilde{\mathcal{D}}_t} \tilde{h}\, d\sigma + \int_{\widetilde{\mathcal{D}}_t\backslash K} \tilde{h}\, d\sigma = \int_{\widetilde{\mathcal{D}}_t} \tilde{h}\, d\sigma = 2\pi \int_{\mathcal{D}_t} h\, dm, \end{aligned} \tag{2.6}$$

where the last equality uses (2.1). Since $\sigma(\widetilde{\mathcal{D}}_t) = 2\pi m(\mathcal{D}_t)$ (again by (2.1)), (2.4) and (2.6) together imply that (2.3), and therefore Theorem 1.1, will be established if we can show that

$$\int_{\mathcal{D}_t} h\, dm \approx m(\mathcal{D}_t)^{1/4}. \tag{2.7}$$

The proof of the estimate (2.7) will be broken down into two lemmas. The first allows us to replace in our estimates the domains $\mathcal{D}_t$, each of which is bounded by an algebraic curve of degree four internally tangent to $\partial\mathbb{D}$ at $\lambda = 1$, by simpler domains.

Lemma 2.1. *There exist constants a, b, c such that if*

$$\xi_t = 1 - at^{-2/3}, \ \eta_t = bt^{-2/3}, \ \text{and } x_t = 1 - ct^{-2/3},$$

then for all t sufficiently large

(i) $\mathcal{D}_t \subset B_t := \{\lambda : \xi_t < \mathrm{Re}(\lambda) < 1, |\mathrm{Im}(\lambda)| < \eta_t\}$;
(ii) $C_t := \{\lambda : |\lambda - x_t| \leq 1 - x_t\} \subset \mathcal{D}_t$;
(iii) $m(\mathcal{D}_t) \approx t^{-4/3}$.

Proof. First note that if we write $\lambda = \xi + i\eta$, $\lambda \in \mathbb{D}$, then for fixed ξ, $1 - |\lambda|^2$ is decreasing in η while $|1-\lambda|^2$ is increasing in η, implying that $h(\xi+i\eta)$ is decreasing in η. Therefore

$$h(\xi + i\eta) \leq h(\xi) = \frac{\sqrt{1-\xi^2}}{(1-\xi)^2} = \frac{\sqrt{1+\xi}}{(1-\xi)^{3/2}} \leq \frac{\sqrt{2}}{(1-\xi)^{3/2}},$$

so that if $\xi < \xi_t$,

$$h(\xi + i\eta) \leq \frac{\sqrt{2}}{(at^{-2/3})^{3/2}} = t$$

if we take $a = 2^{1/3}$. This implies

$$\mathcal{D}_t \subset \mathbb{D} \cap \{\xi + i\eta : \xi \geq \xi_t\}. \tag{2.8}$$

Next, a computation shows that

$$\mathrm{sgn}\frac{\partial}{\partial\xi}h(\xi + i\eta) = \mathrm{sgn}\, G(\xi,\eta),$$

where

$$G(\xi,\eta) = (1-\xi)^2(2+\xi) - \eta^2(2-\xi).$$

Furthermore, assuming $\xi + i\eta \in \mathbb{D}$ and $\xi > 0$, it is easy to check that

$$G(0,\eta) > 0, \ G(\sqrt{1-\eta^2},\eta) < 0, \ \text{ and } \frac{\partial G(\xi,\eta)}{\partial\xi} < 0.$$

It follows that for fixed $\eta \neq 0$, $h(\xi + i\eta)$ attains a maximum on $0 < \xi < \sqrt{1-\eta^2}$ at the unique value $\xi = \xi^*$ satisfying $G(\xi^*, \eta) = 0$, i.e.,

$$(1-\xi^*)^2\kappa = \eta^2, \ \text{where } \kappa = \frac{2+\xi^*}{2-\xi^*}.$$

Substituting this relation into h we get

$$h(\xi^* + i\eta) = \frac{\sqrt{2\kappa - \eta\sqrt{\kappa}(\kappa+1)}}{(\kappa+1)\eta^{3/2}} \leq \frac{\sqrt{2\kappa}}{(\kappa+1)\eta^{3/2}}.$$

Since $0 < \xi^* < 1$, $1 < \kappa < 3$, and so taking $\eta = \eta_t = bt^{-2/3}$ we get

$$h(\xi + i\eta_t) \leq h(\xi^* + i\eta_t) \leq \frac{\sqrt{6}}{2\eta_t^{3/2}} = t$$

if we take $b = (3/2)^{1/3}$. Since h is decreasing in η, this implies

$$\mathcal{D}_t \subset \mathbb{D} \cap \{\xi + i\eta : \eta \leq \eta_t\}. \tag{2.9}$$

Combining (2.8) and (2.9) gives assertion (i).

To prove (ii), note that the fact that $h(\xi + i\eta)$ is decreasing in η implies that on C_t, the minimum of h is attained on ∂C_t. If $|\lambda - x_t| = 1 - x_t$, $\lambda = \xi + i\eta$, it is easy to check that

$$1 - |\lambda|^2 = 2x_t(1-\xi),\ |1-\lambda|^2 = 2(1-x_t)(1-\xi)$$

and therefore for $\lambda \in \partial C_t$,

$$h(\lambda) = \frac{\sqrt{2x_t}}{2(1-x_t)\sqrt{1-\xi}} \geq \frac{\sqrt{2x_t}}{2\sqrt{2}(1-x_t)^{3/2}} \tag{2.10}$$

using the fact that $1 - \xi \leq 2(1 - x_t)$ for $\lambda \in \partial C_t$. If $x_t = 1 - ct^{-2/3}$ with $c = 1/2$, then $x_t > 1/2$ for $t > 1$, and we obtain from (2.10)

$$h(\lambda) \geq \frac{1}{2\sqrt{2}(ct^{-2/3})^{3/2}} = t$$

for $\lambda \in \partial C_t$; this estimate then holds for all $\lambda \in C_t$, by our previous observation. This shows that $C_t \subset \mathcal{D}_t$ and completes the proof of (ii).

Finally, we have shown that $C_t \subset \mathcal{D}_t \subset B_t$ where C_t is a disk with radius $\approx t^{-2/3}$ and B_t is a rectangle with each side of length $\approx t^{-2/3}$. This establishes (iii) and completes the proof of Lemma 2.1. □

Lemma 2.2.

$$\int_{\mathcal{D}_t} h\, dm \approx t^{-1/3}.$$

Proof. First, by Lemma 2.1 (iii),

$$\int_{\mathcal{D}_t} h\, dm \geq \int_{\mathcal{D}_t} t\, dm = tm(\mathcal{D}_t) \gtrsim t^{-1/3}.$$

Next, note that for $t < t', B_{t'} \subset B_t$ and, by Lemma 2.1 (ii), $h < t$ on $\mathbb{D} \setminus B_t$. Extending h to be zero outside the unit disk, we have

$$\begin{aligned}\int_{\mathcal{D}_t} h\, dm \leq \int_{B_t} h\, dm &= \sum_{n=0}^{\infty} \int_{B_{2^n t} \setminus B_{2^{n+1} t}} h\, dm \\ &< \sum_{n=0}^{\infty} 2^{n+1} t \cdot [m(B_{2^n t}) - m(B_{2^{n+1} t})] \lesssim t^{-1/3},\end{aligned}$$

using the fact that $m(B_t) \approx t^{-4/3}$ and summing the series. This completes the proof of Lemma 2.2. □

Finally, the estimate (2.7) follows immediately from Lemma 2.1 (iii) and Lemma 2.2, as does the assertion in Theorem 1.1 that the exponent 1/4 is as large as possible. This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2 and Remarks

To establish Theorem 1.2 fix $K \subset S$ compact, and let Ω be a neighborhood of K in S. Use (1.7) to write for a given $\phi \in C^\infty(S)$,

$$\phi(z) = \Phi(z) + I_1(z) + I_2(z) \tag{3.1}$$

with $\Phi \in A(\mathbb{B})$ and

$$I_1(z) := 2\int_{S\setminus\overline{\Omega}} H(\zeta,z)L\phi(\zeta)d\sigma(\zeta), \quad I_2(z) := 2\int_{\overline{\Omega}} H(\zeta,z)L\phi(\zeta)\, d\sigma(\zeta).$$

Since Φ is a uniform limit on S of functions Φ_n satisfying $L(\Phi_n) = 0, \Phi|_K \in \mathcal{H}(K)$. Moreover, $I_1(z)$ is smooth as a function of z on Ω and by (1.9) satisfies $\overline{L}L(I_1) = 0$ on Ω. Therefore $I_1 \in \mathcal{H}(K)$ and so

$$\text{dist}(\phi, \mathcal{H}(K)) \le \|I_2(z)\|_K. \tag{3.2}$$

For $z \in K$,

$$|I_2(z)| \le 2\|L(\phi)\|_{\overline{\Omega}} \int_{\overline{\Omega}} |H(\zeta,z)|\, d\sigma(\zeta) \le C\|L(\phi)\|_{\overline{\Omega}} \cdot \sigma(\overline{\Omega})^{1/4} \tag{3.3}$$

by Theorem 1.1. Since Ω was an arbitrary neighborhood of K, we may apply (3.3) to a sequence of neighborhoods decreasing to K and conclude from (3.2) that

$$\text{dist}(\phi, \mathcal{H}(K)) \le \|I_2(z)\|_K \le C\|L(\phi)\|_K \cdot \sigma(K)^{1/4}.$$

This completes the proof of Theorem 1.2. □

Remark 3.1. As we noted in the introduction, for a compact plane set K, vanishing of the analytic content $\lambda(K) = \inf\{\|\overline{z} - g(z)\|_K : g \in R(K)\} = 0$ is a necessary and sufficient condition for $R(K)$ to equal $C(K)$, by the Stone-Weierstrass theorem. In ([14]) Khavinson investigated approximation by harmonic functions, replacing the analytic content by

$$\Lambda(K) := \text{dist}(|x|^2, H(K))$$

where for K compact in $\mathbb{R}^n$, $H(K)$ is the closure in $C(K)$ of functions harmonic in a neighborhood of K, and for $x = (x_1, \dots, x_n)$, $|x|^2 = \sum_{j=1}^n x_j^2$. Khavinson proves that $\Lambda(K) = 0$ implies $H(K) = C(K)$ (since $H(K)$ is not an algebra, this does not follow from Stone-Weierstrass), and gives an upper bound on $\Lambda(K)$ in terms of the n-dimensional Lebesgue measure of K. The choice of the function $|x|^2$ is motivated by the fact that the harmonic functions are defined by $\triangle h = 0$, while $\triangle|x|^2$ is constant ($\triangle$ being the Laplace operator). In the setting of rational approximation the functions holomorphic near K (by Runge's theorem, such functions restricted to K belong to $R(K)$) are defined by $\partial g/\partial\overline{z} = 0$, while $\partial\overline{z}/\partial\overline{z} = 1$.

This suggests the question: is there a function $\phi \in C^\infty(S)$ such that for all compact sets K, $\phi \in \mathcal{H}(K)$ implies $\mathcal{H}(K) = C(K)$? Note that $\Box_b\phi = 1$ has no solutions: since $\Box_b$ is self-adjoint as an operator on $L^2(S)$, a necessary condition for the existence of a global solution to $\Box_b\phi = f$ is that f is orthogonal to $\text{Ker}(\Box_b)$. Since $\Box_b(1) = 0$, no such g exists.

Remark 3.2. Khavinson (see [13], also [9]) established the following lower bound on the analytic content of a bounded domain $\Omega \subset \mathbb{C}$ with boundary Γ: $\lambda(\overline{\Omega}) \ge 2m(\Omega)/\ell(\Gamma)$, where $\ell(\Gamma)$ is the length of Γ. Combining this with the upper bound (1.6) applied to $K = \overline{\Omega}$ gives the isoperimetric inequality $\ell(\Gamma)^2 \ge 4\pi m(\Omega)$. For the harmonic content $\Lambda(K)$, as is remarked in ([14]), no such lower bound is possible. In fact, there exist "Swiss cheese" sets K which are the intersection of domains Ω_n with areas $m(\Omega_n)$ bounded away from zero and boundary lengths $\ell(\Gamma_n)$ bounded above, yet $H(K) = C(K)$.

In our setting, a general lower bound on $\operatorname{dist}(\phi, \mathcal{H}(K))$ involving $\|L\phi\|_K$ and/or geometric quantities associated to K is not possible. For example, let

$$K = \{(z_1, z_2) \in S : |z_2| > 1/2\}$$

and let ϕ be a smooth function on S such that $\phi(z) = \overline{z}_1/z_2$ in a neighborhood of K. Then $L(\phi) = 1$ on a neighborhood of K, so $\Box_b \phi = 0$ on a neighborhood of K, i.e., $\phi \in \mathcal{H}(K)$. We may ask if such a lower bound is possible for a particular choice of ϕ. In particular, are there compact subsets of the sphere that are the intersection of domains Ω_n on the sphere for which $\sigma(\Omega_n)$ is bounded away from zero, the two-dimensional Hausdorff measure of $\partial\Omega_n$ is bounded above, and yet $\mathcal{H}(K) = C(K)$?

Remark 3.3. An estimate of the type in Theorem 2 with $\mathcal{H}(K)$ replaced by $R(K)$ or $P(K)$ (the closure in $C(K)$ of the holomorphic polynomials) would be desirable. In particular, such an estimate would settle this open question: does there exist a compact rationally convex (resp. polynomially convex) subset K of S with $\sigma(K) = 0$ but $R(K) \neq C(K)$ (resp. $P(K) \neq C(K)$)? Examples of A. Izzo ([12]) show that such estimates on rational or polynomial approximation cannot hold for compact subsets of the unit sphere in $\mathbb{C}^n$ for any $n > 2$.

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