Approximation Problems on the Unit Sphere in $\mathbb{C}^2$

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Dedicated to the memory of S. Ya. Khavinson

Abstract. Let Γ be the graph of a Hölder continuous function over a Swiss cheese E contained in the open unit disk and having the property that every Jensen measure for $R(E)$ is trivial. We show that if Γ lies in the boundary of the unit ball in $\mathbb{C}^2$, then $R(\Gamma) = C(\Gamma)$. In the appendix we give a geometric interpretation of a class of sets X on the sphere introduced by R. Basener, for which $R(X) \neq C(X)$.

1. Introduction

Let X be a compact subset of $\mathbb{C}^n$. We denote by $R_0(X)$ the algebra of all functions P/Q where P and Q are polynomials on $\mathbb{C}^n$ and $Q \neq 0$ on X, and we denote by $R(X)$ the uniform closure of $R_0(X)$ in the space $C(X)$ of continuous functions on X. We are interested in finding conditions on X that imply that $R(X) = C(X)$, i.e., that each continuous function on X is the uniform limit of a sequence of rational functions holomorphic in a neighborhood of X.

We denote by $h_r(X)$ the rationally convex hull of X, defined as the set of points $y \in \mathbb{C}^n$ such that every polynomial Q with $Q(y) = 0$ vanishes at some point of X. The following is a necessary condition for the equality $R(X) = C(X)$:

$$h_r(X) = X. \tag{1.1}$$

If (1.1) holds, we say that X is *rationally convex*. Rational convexity is invisible when studying rational approximation on plane sets, since every compact plane set satisfies (1.1).

In this article we shall be concerned with the special case of this question when X is a closed subset of the unit sphere $\partial\mathbb{B} = \{(z, w) : |z|^2 + |w|^2 = 1\}$ in $\mathbb{C}^2$. The first result on this problem was obtained by Richard Basener [4] in 1972. Basener constructed a family of rationally convex sets $X_E \subset \partial\mathbb{B}$ for which $R(X) \neq C(X)$. Let E be a compact subset of the open unit disk $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$. For each

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$z \in \mathbb{D}$ we put

$$\gamma_z = \{w \in \mathbb{C} : |z|^2 + |w|^2 = 1\}.$$

Definition 1.1. $X_E = \{(z, w) : z \in E \text{ and } w \in \gamma_z\}$.

Definition 1.2. A *Jensen measure* for a point $z \in E$, relative to the algebra $R(E)$, is a probability measure σ on E such that

$$\log |f(z)| \leq \int_E \log |f| \, d\sigma \quad \text{for all } f \in R(E).$$

For information on Jensen measures, see [6].

Definition 1.3. The set E is of *type* (β) if for all $z \in E$, the only Jensen measure for z relative to $R(E)$ is the point mass δ_z.

Theorem 1.4 (Basener, [4])**.** *Let E be a compact subset of the open unit disk. Assume that $R(E) \neq C(E)$, and that E is of type (β). Then X_E is rationally convex and $R(X_E) \neq C(X_E)$.*

In the converse direction, Basener showed the following (see section 3 of [5]): if X_E is rationally convex, then E is of type (β). We note that if a compact plane set E is of type (β) and $R(E) \neq C(E)$, then E has empty interior and the complement of E is infinitely connected. Sets E with property (β) satisfying $R(E) \neq C(E)$ are known to exist; see the remarks on the "Swiss cheese" sets below.

Corollary 1.5. *Let E be of type (β). Then each closed subset Y of X_E is rationally convex.*

Proof. Fix a point $x \in \mathbb{C}^2 \setminus Y$. If x lies outside X_E, then there exists a polynomial P with $P(x) = 0$ and $P \neq 0$ on X_E, hence $P \neq 0$ on Y. If x lies in X_E, then x belongs to $\partial\mathbb{B} \setminus Y$. It follows that there exists a linear function L with $L(x) = 1$ and $|L| < 1$ on $\partial\mathbb{B} \setminus \{x\}$. Then $L - 1$ vanishes at x but not on Y. □

This corollary provides us with a large collection of rationally convex subsets of $\partial\mathbb{B}$ on which to test the question: what is required of a subset Y of $\partial\mathbb{B}$, beyond rational convexity, in order that the equality $R(Y) = C(Y)$ may hold? We make the following conjecture:

Conjecture: *Let E be a set of type (β) and let f be a continuous complex-valued function defined on E such that $|f(z)| = \sqrt{1 - |z|^2}$ for all $z \in E$. Denote by Γ_f the graph of f in $\mathbb{C}^2$:*

$$\Gamma_f = \{(z, f(z)) : z \in E\}.$$

Then $R(\Gamma_f) = C(\Gamma_f)$.

In Section 2 we shall prove a special case of this conjecture in Theorem 2.1.

Swiss Cheeses: The classical example of a compact plane set E without interior such that $R(E) \neq C(E)$ is the so-called "Swiss cheese" of S. Mergelyan and A.

Roth (see [6]). Fix a closed disk $\overline{D_0} \subset \mathbb{D}$ and choose a countable family of disjoint open disks $D_j, j = 1, 2, \ldots$ contained in D_0, in such a way that

$$E \equiv \overline{D_0} \setminus \bigcup_{j=1}^{\infty} D_j$$

has empty interior. We assume that $\sum_1^\infty r_j < \infty$, where r_j is the radius of D_j. It follows that $R(E) \neq C(E)$ (see [6]). McKissick and others have constructed Swiss cheeses with property (β). At the end of this article, in the Appendix, we show that X_E can be regarded as a three-dimensional Swiss cheese.

2. Rational Approximation on Graphs in $\partial\mathbb{B}$

In our paper [3], entitled "Rational Approximation on the Unit Sphere in $\mathbb{C}^2$," we treated cases of the conjecture stated in the Introduction. To obtain the equality $R(X) = C(X)$ for certain subsets X of $\partial\mathbb{B}$, we imposed on X a strengthening of the rational convexity condition which we called the "hull-neighborhood property" (see Theorem 2.5 of [3]).

It turns out that for a graph Γ_f in $\partial\mathbb{B}$, where f is a function defined on a set of type (β) and satisfying a mild regularity condition, we can dispense with the assumption of the hull-neighborhood condition. We have the following:

Theorem 2.1. *Let E be a compact subset of the open unit disk $\mathbb{D}$ of type (β) and let f be a continuous function on $\mathbb{D}$ satisfying a Hölder condition*

$$|f(z) - f(z')| \leq M|z - z'|^\alpha \quad \textit{for all } z, z' \in \mathbb{D},$$

where M and α are constants, $0 < \alpha < 1$. Assume $|f(z)| = \sqrt{1 - |z|^2}$ for all $z \in \mathbb{D}$. Let Γ_f denote the graph of f over E. Then $R(\Gamma_f) = C(\Gamma_f)$.

Before beginning the proof, we give some preliminaries. Our proof will be based on a transform of measures on $\partial\mathbb{B}$, given by G. Henkin in 1977 in [7], which generalizes the Cauchy transform of measures on plane sets. Let μ be a complex measure on $\partial\mathbb{B}$. In [7], Henkin defined the kernel

$$H(\zeta, z) = \frac{\overline{\zeta_1}\,\overline{z_2} - \overline{\zeta_2}\,\overline{z_1}}{|1 - \langle z, \zeta \rangle|^2} \tag{2.1}$$

on $\partial\mathbb{B} \times \partial\mathbb{B} \setminus \{z = \zeta\}$, where $\langle , \rangle$ denotes the standard Hermitian inner product in $\mathbb{C}^2$. Henkin's transform is the function

$$K_\mu(\zeta) = \int_{\partial\mathbb{B}} H(\zeta, z)\, d\mu(z).$$

Then $K_\mu \in L^1(\partial\mathbb{B})$ and K_μ is smooth on $\partial\mathbb{B} \setminus \operatorname{supp}(\mu)$. (see also [10]).

If μ is orthogonal to polynomials, Henkin showed that

$$\int \phi \, d\mu = \frac{1}{4\pi^2} \int_{\partial\mathbb{B}} K_\mu \, \overline{\partial}\phi \wedge \omega, \quad \text{where } \omega(z) = dz_1 \wedge dz_2 \tag{2.2}$$

for all $\phi \in C^1(\partial\mathbb{B})$. It follows that if X is a closed subset of $\partial\mathbb{B}$ and μ is a measure supported on X with μ orthogonal to $R(X)$, then (2.2) holds. In [9] H.P. Lee and J.Wermer proved that in this setting if X is rationally convex, then K_μ extends from $\partial\mathbb{B} \setminus X$ to the interior of $\mathbb{B}$ as a holomorphic function, again denoted K_μ, by abuse of language.

For each $a \in \mathbb{C}$ we put as earlier

$$\gamma_a = \{w \in \mathbb{C} : |a|^2 + |w|^2 = 1\}$$

and we put

$$\Delta_a = \{(a, w) : |a|^2 + |w|^2 < 1\}.$$

So Δ_a is the disk on the complex line $\{z = a\}$ bounded by the circle $\{(a, w) : w \in \gamma_a\}$. For each a, K_μ restricted to Δ_a is analytic. Without loss of generality we shall assume $E \subset D_0 \equiv \{z : |z| < 1 - \epsilon_0\}$ for some $\epsilon_0 > 0$.

In the proof of Theorem 2.1 we shall make use of the following four results in our paper [3] (proved as Lemma 2.3, Lemma 2.2, Lemma 2.6, and formula (14) of section 4 of that paper, respectively):

Lemma 2.2. *Let μ be a measure on $\partial\mathbb{B}$ and put $X = \operatorname{supp}(\mu)$. Then for all $a \in D$ and for all $w \in \gamma_a$, we have*

$$|K_\mu(a, w)| \leq \frac{4\|\mu\|}{\operatorname{dist}^4((a, w), X)}. \tag{2.3}$$

Here $\|\mu\|$ denotes the total variation of the measure μ. In the next lemma, m_3 refers to three-dimensional Hausdorff measure.

Lemma 2.3. *Let X be a rationally convex subset of $\partial\mathbb{B}$ with $m_3(X) = 0$. Let μ be a measure on X with $\mu \perp R(X)$. If the holomorphic extension of K_μ to B belongs to the Hardy space $H^1(B)$, then $\mu \equiv 0$.*

Lemma 2.4. *With the notations of the preceding Lemma, assume that for some $s > 0$, the restriction of K_μ to Δ_a lies in $H^s(\Delta_a)$ for almost all $a \in D_0$. Then $K_\mu \in H^1(B)$ and so, by Lemma 2.3, $\mu \equiv 0$.*

Lemma 2.5. *Let f and Γ_f be as in Theorem 2.1. Fix a measure μ orthogonal to $R(\Gamma_f)$. There exists a constant c, depending on only on μ, such that for all $a \in D$ and for all $w \in \gamma_a$, we have*

$$|w - f(a)|^{2/\alpha} \leq c \cdot \operatorname{dist}^2((a, w), \Gamma_f). \tag{2.4}$$

We are now ready to begin the proof of Theorem 2.1.

Lemma 2.6. *Let f be as in Theorem 2.1 and let μ be a measure on Γ_f orthogonal to $R(\Gamma_f)$. There exists a constant κ depending only on μ such that for all $a \in \overline{D_0}$, setting $r = \sqrt{1 - |a|^2}$, we have*

$$\int_0^{2\pi} |K_\mu(a, re^{i\phi})|^{\alpha/8}\, d\phi \leq \kappa. \tag{2.5}$$

Proof. Fix $a \in \overline{D_0}, \phi \in [0, 2\pi]$ and put $w = re^{i\phi}$. By (2.4),

$$\frac{1}{\operatorname{dist}^4((a,w),\Gamma_f)} \leq \frac{c^2}{|w - f(a)|^{4/\alpha}}.$$

The estimate (2.3) then gives

$$|K_\mu(a,w)| \leq \frac{4\|\mu\|c^2}{|w - f(a)|^{4/\alpha}}$$

and so

$$|K_\mu(a,w)|^{\alpha/8} \leq \frac{c_2}{|w - f(a)|^{1/2}},$$

where c_2 is a constant depending only on μ. Thus

$$\int_0^{2\pi} |K_\mu(a, re^{i\phi})|^{\alpha/8}\, d\phi \leq c_2 \int_0^{2\pi} \frac{d\phi}{|re^{i\phi} - f(a)|^{1/2}}.$$

We write $f(a) = re^{i\phi_0}$. Then the right-hand side equals

$$c_2 \int_0^{2\pi} \frac{d\phi}{r^{1/2}|e^{i\phi} - e^{i\phi_0}|^{1/2}} \leq \frac{c_2}{r^{1/2}} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - 1|^{1/2}}.$$

Note that the integral on the right hand side of the last inequality is finite. Also, since $|a| < 1 - \epsilon_0$, there exists $r_0 > 0$ such that $r > r_0$ for every $a \in \overline{D_0}$. So

$$|K_\mu(a,w)|^{\alpha/8} \leq \frac{c_2}{r_0^{1/2}} \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - 1|^{1/2}}.$$

Denoting this last expression by κ, we get (2.5). □

Lemma 2.7. *Fix $a \in \overline{D_0} \setminus E$. Put $r = \sqrt{1 - |a|^2}$. For $R < r$ we have*

$$\int_0^{2\pi} |K_\mu(a, Re^{i\theta})|^{\alpha/8}\, d\phi \leq \kappa, \tag{2.6}$$

where κ is the constant in (2.5).

Proof. Since a lies outside E, $\overline{\Delta_a}$ is disjoint from Γ_f, so the restriction of K_μ to Δ_a extends continuously to $\overline{\Delta_a}$. It is well known that the function

$$R \to \int_0^{2\pi} |K_\mu(a, Re^{i\phi})|^{\alpha/8}\, d\phi$$

is monotonic on $0 < R < r$ and continuous on $0 \leq R \leq r$. So (2.5) implies (2.6). □

Lemma 2.8. *Fix a_0 in E. Fix $R < \sqrt{1 - |a_0|^2}$. Then*

$$\int_0^{2\pi} |K_\mu(a, Re^{i\theta})|^{\alpha/8}\, d\phi \leq \kappa, \tag{2.7}$$

where κ is the constant in (2.5).

Proof. Choose a sequence $\{a_n\}$ converging to a_0 such that $a_n \in D_0 \setminus E$ for each n. For n large, then, $R < \sqrt{1-|a_n|^2}$. By Lemma 2.7,

$$\int_0^{2\pi} |K_\mu(a_n, Re^{i\theta})|^{\alpha/8}\, d\phi \leq \kappa, \ \ n \gg 1.$$

Also $K_\mu(a_n, Re^{i\theta}) \to K_\mu(a_0, Re^{i\theta})$ uniformly on $0 \leq \phi \leq 2\pi$ as $n \to \infty$ since K_μ is continuous on $\operatorname{int}(B)$. By continuity, then, we get (2.7). □

Lemmas 2.7 and 2.8 say that for all $a_0 \in \overline{D_0}$, K_μ restricted to Δ_{a_0} lies in $H^{\alpha/8}(\Delta_{a_0})$. Lemma 2.4 then yields that $\mu \equiv 0$. Since this holds for each μ orthogonal to $R(\Gamma_f)$, we conclude that $R(\Gamma_f) = C(\Gamma_f)$, and so Theorem 2.1 is proved. □

Appendix: Geometric Interpretation of the sets X_E

We shall show that the sets X_E lying in $\partial\mathbb{B}$ can be seen as three-dimensional analogues of the Swiss cheese E in $\mathbb{C}$. We denote k-dimensional Hausdorff measure by m_k.

The Swiss cheese E is constructed by removing a countable family of disjoint open disks D_j from a closed disk $\overline{D_0}$. The following properties hold:

- **(i):** $\sum_{j=1}^{\infty} m_1(\partial D_j) < \infty$;
- **(ii):** $m_2(E) > 0$;
- **(iii):** The measure dz restricted to the union of the circles ∂D_j (properly oriented) is finite on E and is orthogonal to $R(E)$.

It follows immediately from (iii) that $R(E) \neq C(E)$.

Let us now start with a family of disks D_j in $\mathbb{C}$ as above and let us replace each D_j by the open solid torus $T_j = \{(z,w) \in \partial\mathbb{B} : z \in D_j\}, j = 1, 2, \ldots$, with $T_0 = \{(z,w) \in \partial\mathbb{B} : z \in D_0\}$. Set

$$E^* = \overline{T_0} \setminus \bigcup_{j=1}^{\infty} T_j.$$

Then E^* is a compact subset of $\partial\mathbb{B}$ with the following properties:

- **(i′) :** $\sum_{j=1}^{\infty} m_2(\partial T_j) < \infty$;
- **(ii′) :** $m_3(E^*) > 0$;
- **(iii′) :** The measure $\mu = dz \wedge dw$ restricted to the union of the boundaries ∂T_j (properly oriented) is finite on E^* and is orthogonal to $R(E^*)$.

Properties (i′) and (ii′) follow immediately from Fubini's Theorem and properties (i) and (ii) of the Swiss Cheese. As for (iii′), the finiteness of $\mu = dz \wedge dw$ follows from assumption (i′) together with the following assertion.

Claim: Let M be a smooth two (real) dimensional submanifold of $\mathbb{C}^2$, S a Borel subset of M, and m_2 two-dimensional Hausdorff measure. Then

$$\|\mu\|(S) \leq m_2(S).$$

Here $\|\mu\|$ denotes the total variation measure of μ.

Proof. Identify $\mathbb{C}^2$ with $\mathbf{R}^4$ using coordinates

$$z = x + iy,\ w = u + iv.$$

We may assume that near S, M is given parametrically, i.e., is the image of a smooth map Φ from a neighborhood of the origin in $\mathbf{R}^2$ to M. Using coordinates (ξ, η) in $\mathbf{R}^2$, let E_1, E_2 be the images of the tangent vectors $\partial/\partial\xi$ and $\partial/\partial\eta$ under the differential of Φ, so

$$E_1 = (x_\xi, y_\xi, u_\xi, v_\xi),\quad E_2 = (x_\eta, y_\eta, u_\eta, v_\eta)$$

as vectors in $\mathbf{R^4}$, where subscripts denote partial derivatives. It is standard that the two-dimensional volume form on M is given by (the area of the parallelogram spanned by E_1, E_2):

$$dV = \sqrt{\det(g)}\ d\xi d\eta,$$

where g is the 2×2 matrix with entries $g_{ij} = E_i \cdot E_j$, $i, j = 1, 2$ and $\cdot$ is the usual inner product in $\mathbf{R}^4$. It is also well-known that

$$m_2(S) = \int_S dV.$$

On the other hand, writing $dx = x_\xi d\xi + x_\eta d\eta$, etc., we obtain

$$dz \wedge dw = (dx + idy) \wedge (du + idv) = (A + iB)\ d\xi \wedge d\eta,$$

where

$$A = x_\xi u_\eta - x_\eta u_\xi - y_\xi v_\eta + y_\eta v_\xi$$

and

$$B = x_\xi v_\eta - x_\eta v_\xi + y_\xi u_\eta - y_\eta u_\xi.$$

To establish the claim it suffices to show that

$$\det(g) \geq A^2 + B^2. \tag{2.8}$$

A calculation gives

$$\det(g) - (A^2 + B^2) = (x_\xi y_\eta - x_\eta y_\xi + v_\eta u_\xi - v_\xi u_\eta)^2,$$

which establishes (2.8) and completes the proof of the claim. □

To prove the assertion of (iii′) that $dz \wedge dw$ is orthogonal to $R(E^*)$, we argue as follows: fix a rational function $f = P/Q$, where P, Q are polynomials with $Q \neq 0$ on E^*. The set $\{Q = 0\} \cap \overline{T_0}$ is contained in $\bigcup_{j=1}^\infty T_j$. By Heine-Borel, there exists an integer such that this set is contained in $\bigcup_{j=1}^n T_j$. We put $\Omega_n = T_0 \setminus \bigcup_{j=1}^n T_j$. Then f is holomorphic on $\overline{\Omega_n}$. By Stokes' Theorem applied to the form $f dz \wedge dw$ on Ω_n, we have

$$\int_{\partial\Omega_n} f dz \wedge dw = \int_{\Omega_n} \bar{\partial} f \wedge dz \wedge dw.$$

The right-hand side of this equation vanishes, since f is analytic on a neighborhood of $\overline{\Omega_0}$. The left-hand side approaches $\int_{E^*} f\ d\mu$ as $n \to \infty$. So $\int_{E^*} f\ d\mu = 0$. Thus

μ is orthogonal to f. Since this holds for each $f \in R_0(E^*)$, we have μ orthogonal to $R(E^*)$.

Finally, we remark that $R(E^*) \neq C(E^*)$ clearly follows from (iii$'$). It is clear that E^* coincides with X_E, by the definition of X_E in the Introduction. For an arbitrary Swiss cheese E, X_E will not be rationally convex.

There is a substantial literature related to the approximation questions treated in this article. The references below list some of the relevant papers, as well as those papers specifically cited in this article.

References

[1] J. Anderson and J. Cima, *Removable Singularities for L^p CR functions*, Mich. Math. J. **41** (1994), pp. 111-119.

[2] J. Anderson and A. Izzo, *A Peak Point Theorem for Uniform Algebras Generated by Smooth Functions On a Two-Manifold*, Bull. London Math. Soc. **33** (2001), pp. 187–195.

[3] J. Anderson, A. Izzo and J. Wermer, *Rational Approximation on the Unit Sphere in $\mathbb{C}^2$*, to appear in Mich. J. Math

[4] R. F. Basener, *On Rationally Convex Hulls*, Trans. Amer. Math. Soc. **182** (1973), pp. 353–381.

[5] R. F. Basener, *Rationally Convex Hulls and Potential Theory*, Proc. Royal Soc. Edinburgh, **74A** (1974/5), pp. 81-89.

[6] T. Gamelin, *Uniform Algebras*, 2nd ed. Chelsea, New York, 1984.

[7] G. M. Henkin, *The Lewy equation and analysis on pseudoconvex manifolds*, Russian Math. Surveys, **32:3** (1977); Uspehi Mat. Nauk **32:3** (1977), pp. 57–118.

[8] H. P. Lee, *Tame measures on certain compact sets*, Proc. A. M. S. (1980) pp. 61-67

[9] H. P. Lee and J. Wermer, *Orthogonal Measures for Subsets of the Boundary of the Ball in $\mathbb{C}^2$*, in *Recent Developments in Several Complex Variables*, Princeton University Press, 1981, pp. 277-289.

[10] N. Th. Varopoulos, *BMO functions and the $\overline{\partial}$-equation*, Pac. J. Math. **71**, no. 1 (1977), pp. 221–273.

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